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FRANK MORLEY, EDITOR

A. COHEN, ASSISTANT EDITOR

WITH THE COÖPERATION OF

CHARLOTTE A. SCOTT, A. B. COBLE

AND OTHER MATHEMATICIANS

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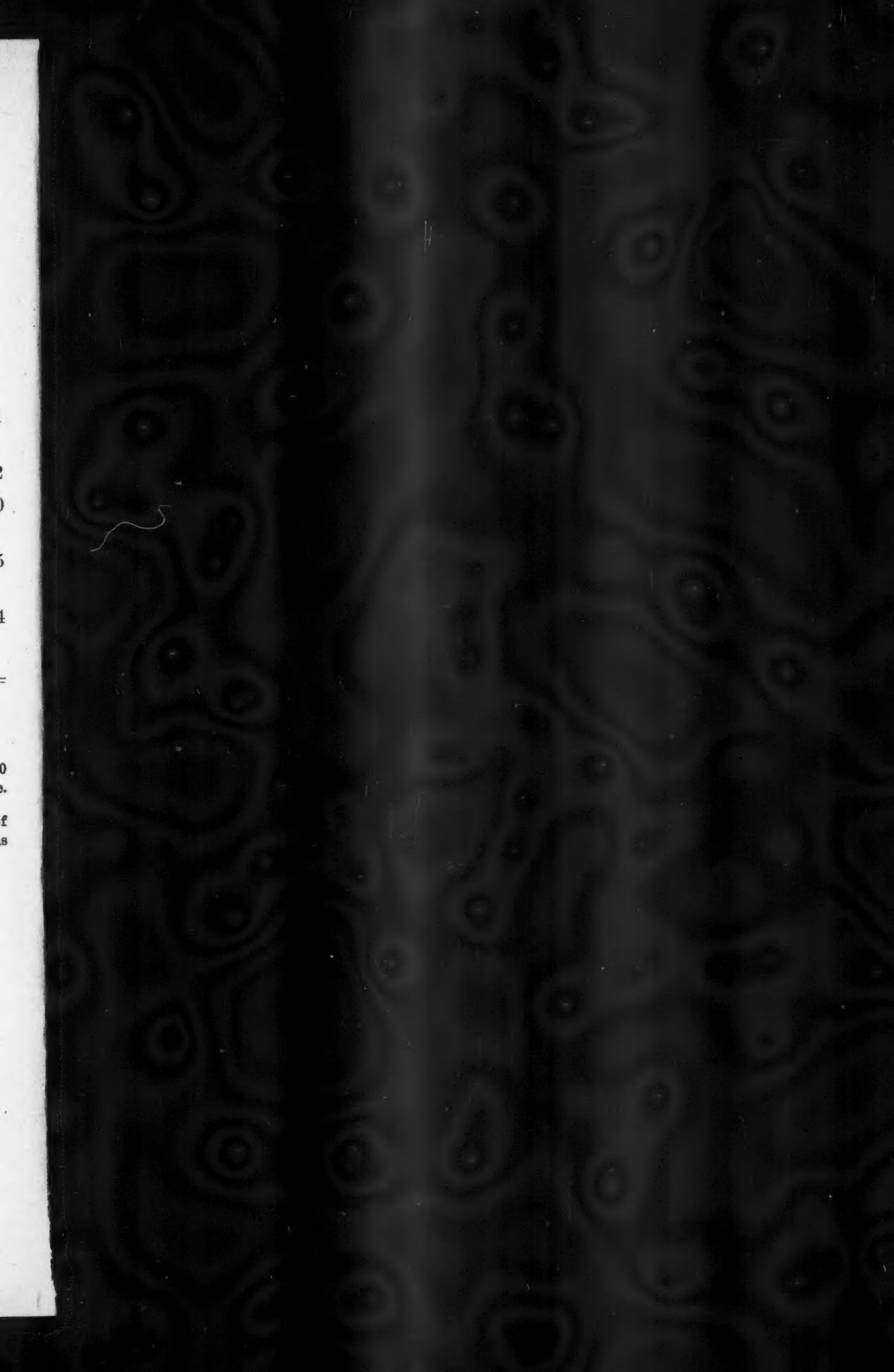

AN ARITHMETICAL DUAL OF KUMMER'S QUARTIC SURFACE.*

By E. T. Bell.

1. We shall use the well-established notations $\alpha \supset \beta$ to signify that α implies β, and $\alpha \equiv \beta$ to express the formal equivalence of α and β, viz. $\alpha \supset \beta$ *and* $\beta \supset \alpha$. Consider the homogeneous algebraic equation

$$\Phi(x, y, z, w) = 0. \tag{A}$$

When x, y, z, w are the homogeneous coördinates of a point, (A) is the equation S' of a surface S. If x, y, z, w are assigned any other interpretation I, (A) is then called the equation E' of the equivalent E of S with respect to I. If $S' \equiv E'$, we shall call S, E duals of each other with respect to I; and if I is arithmetical, E is defined to be an arithmetical dual of S.

Let E be an arithmetical dual of S, and suppose that all the properties of integers implicit in E are transposed into a set P of properties of point configurations lying in a lattice space of the appropriate number of dimensions.† If now $P \supset E'$, then E, S being arithmetical duals, $P \supset S'$; and conversely $S' \supset P$. Hence since $P \equiv S'$ we may regard S in continuous point space as the equivalent or image of P' in the discrete lattice space. Another image of S will be glanced at in § 10. Clearly all of these considerations can be readily modified to hold for any system of equations representing loci in any space.

2. The most interesting of these duals appear to be those of the non-rational plane algebraic curves and the hyperelliptic r-folds in space of $r(r+1)/2$ dimensions. Obvious geometrical properties suggest analytical relations which dualize into interesting properties of numbers having many applications to the arithmetic of systems of higher forms. Thus we have a new application of geometry to numbers.

Here we briefly consider an arithmetical dual of the quartic surface with sixteen nodes, carrying the development up to the point of providing an immediate means for translating all the geometry of the surface directly

* Read before the San Francisco Section of the American Math. Soc., June 17, 1920.

† The E of Kummer's S considered in this paper can be interpreted in terms of lattice configurations lying upon two concentric spheres in a space of sixteen dimensions, but this is not developed here. This interpretation will be sufficiently evident from § 7 on applying the isomorphism there established to the rationalized form of the equation of S as given in Borchardt, *Crelle*, 83 (1877), p. 238; or Krause, *Die Transf. Hyperell. Funkt. erster Ord.* (Leipsig, 1886), p. 39; or Hudson, *Kummer's Quartic Surface* (Cambridge, 1905), p. 81. Rohn's parametric representations of S, *Math. Annalen*, 15 (1874), p. 315, based upon the transformation of the second order, give two more arithmetical duals distinct from E.

Bell

into arithmetic, the translation being reversible, so that from it the geometry can be recovered. Thus, for example, the 16_6 configuration of nodes and tropes is equivalent to a remarkable interlacing of properties of sets of three integers constructed from the decompositions of a pair of arbitrary integers into two sums of four squares, and conversely these imply the existence of the configuration.

3. Henceforth all letters l, m, n denote integers $\gtreqless 0$ unless further specified; *the l's are always even and the m's odd.* Write*

$$[\sum_{j=1}^{r} n_j] \equiv [n_1, n_2, \cdots, n_r],$$

and let either of these symbols denote $1, i, -1, -i$ $(i \equiv \sqrt{-1})$ according as

$$\sum_{j=1}^{r} n_j \equiv 0, 1, 2, 3 \bmod 4.$$

From the definition we have

$$[n_1, n_2, \cdots, n_r] = \prod_{j=1}^{r} [n_j]; \qquad [l] = (-1)^{l/2}; \qquad [m] = i(-1)^{(m-1)/2},$$

so that the symbol represents a real or imaginary unit according as the number of odd n_j is even or odd. We define $F(x, y, z)$ to be an arithmetical function if it exists and is single valued for x, y, z simultaneously integers $\gtreqless 0$, and otherwise is wholly arbitrary. If further conditions are imposed upon F it is called restricted; and unless the contrary is expressly noted, throughout the paper $F(x, y, z)$ denotes an unrestricted arithmetical function. An immediate consequence of this definition is of importance later (§ 10).

If

$$\sum_{j=1}^{r} F(n_{1j}, n_{2j}, n_{3j}) = \sum_{k=1}^{s} F(n_{1k}', n_{2k}', n_{3k}'),$$

then $r = s$, and in some order the triads (n_{1j}, n_{2j}, n_{3j}) are identical with the triads $(n_{1k}', n_{2k}', n_{3k}')$, two such being identical when and only when* $n_{ij} = n_{ik}'$ $(i = 1, 2, 3)$.

4. Let g_1, g_2, h_1, h_2 denote constant integers $\geqq 0$, and write $2n_j + g_j \equiv \nu_j$ $(j = 1, 2)$. Then the *general f-function* is defined by

$$f(\nu_1, \nu_2) \equiv f\begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(n_1, n_2)} = [h_1\nu_1, h_2\nu_2] F(\nu_1, \nu_2, \nu_1\nu_2).$$

* The $\equiv$ here means algebraic identity. There can be no confusion between this and the sign of formal equivalence, as in each case the context indicates which is meant. The [ ] notation is merely to avoid the printing of complicated exponents.

† If a formal proof of this proposition is desired, it can easily be constructed from section II of "Arithmetical Paraphrases," to appear shortly in the *Trans. Am. Math. Soc.*

The symbol $\begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}$ is the *mark* of this f, and is even or odd according as $g_1h_1 + g_2h_2$ is even or odd. According as its mark is even or odd, f is *quasi-even* or *quasi-odd*. If the f just written is quasi-even, the function

$$\varphi(\nu_1, \nu_2) \equiv \varphi \begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(n_1, n_2)} = [h_1\nu_1, h_2\nu_2]F(0, 0, \nu_1\nu_2)$$

is the *quasi-constant* corresponding to f. The integers ν_1, ν_2 are the *parameters of f or φ*.

For g_1, g_2, h_1, $h_2 = 0, 1$ there are thus 16 marks, of which 10 are even and 6 odd. Corresponding to these marks are 16 distinct f-functions which fall into sets of 4 each according to the residues mod 2 of the variables ν_1, ν_2:

$$(\nu_1, \nu_2) = (l_1, l_2), (m_1, l_2), (l_1, m_2), (m_1, m_2).$$

Transposing the entire theory of double theta marks to the present subject, we name each reduced mark, excluding (00), by the particular dyad of odd marks to which it is congruent.* If now (ij) is any one of the 16 reduced marks, (00) included, the $f(\nu_1, \nu_2)$ whose mark is (ij) is denoted by $f_{ij}(\nu_1, \nu_2)$, or simply by f_{ij}; and if (ij) is even, the corresponding quasi-constant is $\varphi_{ij}(\nu_1, \nu_2)$ or φ_{ij}. In this notation the order of i, j in the suffix is indifferent. We thus have the following system of 16 f-functions in which the parity of the variables is indicated by the l, m notation (§ 3):

$$\begin{array}{llll}
f_{00} = F(l_1, l_2, l_1l_2), & f_{23} = [l_1]f_{00}, & f_{45} = [l_2]f_{00}, & f_{16} = [l_1, l_2]f_{00}; \\
f_{12} = F(m_1, l_2, m_1l_2), & f_{13} = [m_1]f_{12}, & f_{36} = [l_2]f_{12}, & f_{26} = [m_1, l_2]f_{12}; \\
f_{56} = F(l_1, m_2, l_1m_2), & f_{14} = [l_1]f_{56}, & f_{46} = [m_2]f_{56}, & f_{15} = [l_1, m_2]f_{56}; \\
f_{34} = F(m_1, m_2, m_1m_2), & f_{24} = [m_1]f_{34}, & f_{35} = [m_2]f_{34}, & f_{25} = [m_1, m_2]f_{34}.
\end{array}$$

The six quasi-odd functions are

$$f_{13}, f_{24}, f_{46}, f_{35}, f_{26}, f_{15};$$

and the ten quasi-constants are

$$\varphi_{00}, \varphi_{12}, \varphi_{56}, \varphi_{34}, \varphi_{23}, \varphi_{14}, \varphi_{45}, \varphi_{36}, \varphi_{16}, \varphi_{25}.$$

Henceforth in all functions of f's and φ's, the φ's are regarded as being of degree zero, so that *in determining the degree of any expression involving f's and φ's the φ's are to be considered as absolute constants.* Thus $\varphi_{13}^2 f_{24}^2 + \varphi_{35}^2 f_{15}^2 = 0$ is a homogeneous linear relation between squares of f-functions.

5. The arithmetic-geometric translations mentioned in § 2 depend upon

* See Harkness and Morley, "Treatise on the Theory of Functions," ch. 8, whose notation we shall follow throughout; also Weber, "Über die Kummersche Fläche," u.s.w., *Crelle*, 84 (1878), especially pp. 332–334.

a form of symbolic product, or henceforth simply product, of $(r-s)$ f-functions and s quasi-constants, $0 \leqq s \leqq r$. These products are taken with respect to a pair of integers $n_1,\ n_2 > 0$ of preassigned linear forms mod 4:

$$n_j \equiv t_j \bmod 4;\ t_j \text{ constant}, \quad (j = 1, 2).$$

Let all the $g,\ h$ denote constants, each of which is one of the numbers 0, 1. Write

$$\begin{bmatrix} g_{1i}, & g_{2i} \\ h_{1i}, & h_{2i} \end{bmatrix} \equiv (i), \qquad 2n_{ji} + g_{ji} \equiv \nu_{ji} \quad (j = 1, 2),$$

and suppose that precisely t_1 of the g_{1i} each $= 1$, so that precisely t_1 of the ν_{1i} are odd, and similarly for t_2 and the $g_{2i},\ \nu_{2i}$. The f's or φ's for the marks (i), (k) are

$$\begin{aligned} f_i(\nu_{1i},\ \nu_{2i}) \equiv f_i &= [h_{1i}\ \nu_{1i},\ h_{2i}\nu_{2i}]F(\nu_{1i},\ \nu_{2i},\ \nu_{1i}\nu_{2i}), \\ \varphi_k(\nu_{1k},\ \nu_{2k}) \equiv \varphi_k &= [h_{1k}\nu_{1k},\ h_{2k}\nu_{2k}]F(0,\ 0,\ \nu_{1k}\nu_{2k}), \\ &\qquad (i = s+1,\ s+2,\ \cdots,\ r;\ k = 1, 2, \cdots, s). \end{aligned}$$

Put

$$h \equiv \sum_{i=1}^{r} (h_{1i}\nu_{1i} + h_{2i}\nu_{2i}), \quad \xi_{12} \equiv \sum_{i=1}^{r} \nu_{1i}\nu_{2i}, \quad \xi_j \equiv \sum_{j=s+1}^{r} \nu_{ji} \quad (j = 1, 2).$$

Then, the Σ extending to all $\nu_{ji} \gtrless 0$ such that

$$n_j = \nu_{j1}^2 + \nu_{j2}^2 + \cdots + \nu_{jr}^2 \quad (j = 1, 2),$$

the *product with respect to* $n_1,\ n_2$ of the s quasi-constants and $(r-s)$ f-functions just written is defined by

$$\prod_{i=1}^{s}{}' \varphi_i(\nu_{1i},\ \nu_{2i}) \cdot \prod_{i=s+1}^{r}{}' f_i(\nu_{1i},\ \nu_{2i}) \equiv \Sigma[h]F(\xi_1,\ \xi_2,\ \xi_{12}),$$

and the *type** of this product is $\{t_1, t_2\}$, $\equiv \{t_2, t_1\}$. The accent in Π' indicates that the multiplication is purely symbolic.

Note that the sum on the right consists of only a finite number of terms. For if $N_r(n,\ s)$ is the total number of representations of n as a sum of r squares precisely s of which are odd and occupy fixed positions, the number of terms is $N_r(n_1,\ t_1)N_r(n_2,\ t_2)$. Note also (cf. § 3) that if $t_1 + t_2$ is even, $[h]$ is real. In the products connected with Kummer's surface it will be seen at once that each $f_i,\ \varphi_k$ occurs an even number of times. Hence in these products $[h]$ is always real, and we need not again refer to this.

When a_j of the marks (1), (2), $\cdots$, (s) each $= j$, and b_k of the marks $(s+1)$, $(s+2)$, $\cdots$, (r) each $= k$, the product is written $\varphi_1{}^{a_1}\varphi_2{}^{a_2} \cdots f_1{}^{b_1}f_2{}^{b_2} \cdots$; and clearly the order of the factors $\varphi_1{}^{a_1}$, $\cdots$, $f_1{}^{b_1}$, $\cdots$ in this is im-

* The type is not important for this paper, and is included here merely for completeness. It plays an essential part in the detailed discussion of the arithmetical nodes and tropes on E, cf. § 9, second footnote.

material. Hence such multiplication of φ's and f's is commutative. We shall not be concerned here with its associative and distributive aspects. By definition $\varphi_i^0, f_i^0 = 1$; and unity in this multiplication is to have all the formal properties of unity in ordinary multiplication, so that unit factors may be suppressed.

The *parameters of the product* are n_1, n_2. These ultimately play a part, in one geometrical interpretation of the arithmetic, analogous to that of v_1, v_2, the parameters of a point on the Kummer surface; cf. § 9. Any product p_i being a sum with respect to given parameters, the meaning of $\Sigma_i a_i p_i$ where the a_i are absolute constants is evident. Note that

$$a_i p_i + a_j p_i = (a_i + a_j) p_i$$

when and only when both p_i are with respect to the same parameters. To indicate that each p_i in the sum is with respect to n_1, n_2 we write $n_1, n_2 \,|\, \Sigma_i a_i p_i$. If now

$$n_1, n_2 \,|\, \sum_i a_i p_i = 0, \tag{1}$$

it follows at once from the definitions that we may replace n_j by n_j' where $n_j' \equiv t_j \bmod 4$ $(j = 1, 2)$. For each pair (n_1', n_2'), (1) gives a relation

$$n_1', n_2' \,|\, \sum_i a_i p_i = 0 \tag{2}$$

between products. To signify that we are considering the totality of relations of the form (2) for all (n_1', n_2'), we shall write (1) without the $n_1, n_2 \,|$, thus,

$$\Sigma a_i p_i = 0. \tag{3}$$

It is important to note that (3) is of the form $P(f, \varphi) = 0$, where $P(f, \varphi)$ is a polynomial in f's and φ's.

Consider now $R(f, \varphi) = 0$ where $R(f, \varphi)$ is a rational algebraic function of f's and φ's. As yet this relation has no significance. By purely formal algebraic reductions $R(f, \varphi) = 0$ may be written $P(f, \varphi) = 0$, $P(f, \varphi)$ as above, and we define this to be the meaning of $R(f, \varphi) = 0$, again emphasizing the significance of the omission of $n_1, n_2 \,|$. Thus in all that follows, $R(f, \varphi) = 0$ and $P(f, \varphi) = 0$, its polynomial equivalent, are regarded as identical.

6. Denote by $R(\theta, c) = 0$ a rational algebraic relation with rational coefficients $\gtrless 0$ between the sixteen double theta functions θ_{ij} and the related constants* c_{ij}.

* The notation is that of Harkness and Morley, loc. cit. The restriction that R be rational is merely to shorten the following proof. By a few simple changes the theorem may be restated for R algebraic with algebraic number coefficients, but this is not required for the dual of Kummer's surface.

By algebraic reductions $R(\theta, c) = 0$ may be cast in the form $P(\theta, c) = 0$, where now P is a polynomial with integral coefficients $\gtreqless 0$. We shall regard $R(\theta, c) = 0$, $P(\theta, c) = 0$ as identical statements. In these replace θ_{ij}, c_{ij} by f_{ij}, φ_{ij} respectively, and denote the results by $R(f, \varphi) = 0$, $P(f, \varphi) = 0$. It is not immediately evident, of course, that $P(f, \varphi) = 0$, or its formal equivalent $R(f, \varphi) = 0$, is true. The fundamental theorem is:

$$\{P(\theta, c) = 0\} \equiv \{P(f, \varphi) = 0\},$$

(the sign $\equiv$ being, as in the next section, that defined in § 1), or what is the same thing,

$$\{R(\theta, c) = 0\} \equiv \{R(f, \varphi) = 0\}.$$

No doubt it is easy to prove this from first principles. It is less tedious, however, to proceed as follows.

7. Clearly $P(f, \varphi) = 0$ is a finite sum relation of the form

$$\sum_k a_k F(\alpha_k, \beta_k, \gamma_k) = 0,$$

in which a_k, α_k, β_k, γ_k are integers, and it is easily seen that if $g(x, y, z)$ is any restricted arithmetical function such that $g(x, y, z) = -g(-x, -y, -z)$, then $\Sigma \pm g(\pm x, \pm y, \pm x \times \pm y)$ is not identically zero. In each f, φ replace $F(\alpha_k, \beta_k, \gamma_k)$ by $\sin(\alpha_k x + \beta_k y + \gamma_k z)$, where x, y, z are parameters, and denote by $P_1(f, \varphi) = 0$ what $P(f, \varphi) = 0$ becomes under this substitution; and similarly for $F(\alpha_k, \beta_k, \gamma_k)$ replaced by $\cos(\alpha_k x + \beta_k y + \gamma_k z)$, giving $P_2(f, \varphi) = 0$. Write $P_3(f, \varphi) = iP_1(f, \varphi) + P_2(f, \varphi)$, $(i = \sqrt{-1})$.

We shall be concerned with the implications of the five propositions defined by

$\alpha \equiv \{P(f, \varphi) = 0\}$, $\beta \equiv \{R(\theta, c) = 0\}$, $\alpha_j \equiv \{P_j(f, \varphi) = 0\}$, $(j = 1, 2, 3)$, and will show that

(4) $\alpha \supset \alpha_3$, (6) $\alpha_3 \supset \beta$,
(5) $\beta \supset \alpha_3$, (7) $\alpha_3 \supset \alpha$;

whence from (4), (6) it will follow that $\alpha \supset \beta$, and from (5), (7) that $\beta \supset \alpha$, and therefore from these $\alpha \equiv \beta$, which is the theorem.

From the definition of F (§ 3), (4) is obvious. To prove (5) and (6) we observe that $P_3(f, \varphi)$ is the coefficient of $p_1^{n_1} p_2^{n_2}$ in $R(\theta, c) = 0$ when the expansion of the general theta function

$$\theta \begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}^{(x/\pi,\, y/\pi,\, 4\tau_{11},\, 2z/\pi,\, 4\tau_{22})}, \quad (z = \pi\tau_{12}/2),$$

is written in the form

$$\Sigma p_1^{\nu_1^2} p_2^{\nu_2^2} [h_1\nu_1, h_2\nu_2] \exp i(\nu_1 x + \nu_2 y + \nu_1\nu_2 z), \quad (p_j = \exp i\pi\tau_{jj}),$$

the Σ referring to all $\nu_j = 2n_j + g_j$ $(j = 1, 2)$.

To prove (7) consider two restricted arithmetical functions (cf. § 3) ψ_1, ψ_2, the only restrictions upon them being

$$\psi_j(u, v, w) = (-1)^j\psi_j(-u, -v, -w), \quad (j = 1, 2),$$

and consider the propositions α_j' defined by the formal equivalences

$$\alpha_j' \equiv \{\sum_k a_k\psi_j(\alpha_k, \beta_k, \gamma_k) = 0\}, \quad (j = 1, 2).$$

Then it may be shown without difficulty* that

$$\alpha_j \supset \alpha_j', \quad (j = 1, 2). \tag{8}$$

Now choose for the ψ_j the restricted arithmetical functions defined by

$$2\psi_j(u, v, w) = F(u, v, w) + (-1)^jF(-u, -v, -w), \quad (j = 1, 2),$$

which obviously satisfy the restrictions imposed upon the ψ_j. Substitute these values of the ψ_j in the α_j' respectively, getting α_j''. Then from (8)

$$\alpha_j \supset \alpha_j'', \quad (j = 1, 2). \tag{9}$$

If now $i\alpha_1$ denotes the result of replacing $\sin(\alpha_k x + \beta_k y + \gamma_k z)$ in α_1 by i times itself, we have $i\alpha_1 \supset \alpha_1$, and hence from (9)

$$i\alpha_1 \supset \alpha_1'' \quad \textit{and} \quad \alpha_2 \supset \alpha_2''. \tag{10}$$

But obviously

$$(\alpha_1'' \textit{ and } \alpha_2'') \supset \alpha, \text{ also } (i\alpha_1 \textit{ and } \alpha_2) \equiv \alpha_3;$$

while from (10),

$$(i\alpha_1 \textit{ and } \alpha_2) \supset (\alpha_1'' \textit{ and } \alpha_2'').$$

Hence $\alpha_3 \supset \alpha$, which is (7).

8. The theorem of § 6 enables us to transpose the entire theory of the rational algebraic relations between the θ_{ij}, c_{ij} to the present subject. In particular there is here a system of 16 Rosenhain hexads of which one is†

$$f_{13}, f_{35}, f_{15}, f_{24}, f_{46}, f_{26}$$

such that there is a linear relation (cf. § 4 end) between the squares of any four f-functions belonging to the same hexad; also the sixteen f_{ij} fall into 60 Göpel tetrads, the four f-functions in a tetrad being connected by a homogeneous biquadratic relation. Again, between the squares of any five f-functions, no four of which belong to a hexad, there is a linear relation; and we may state the general theorem* that the squares of four f-functions are linearly related when and only when their marks belong to a Rosenhain

* This is included as a very special case of a theorem proved in "Arithmetical Paraphrases," Part I, Sec. II, cited in § 3, footnote.

† The complete system may be written down from the table in Harkness and Morley, loc. cit., pp. 353–354.

‡ Harkness and Morley, loc. cit., p. 368.

hexad, and the squares of any five f-functions no four of which are in one hexad are linearly related. All of these results have immediate arithmetical interpretations in terms of representations of n_1, n_2 either as sums of four squares (for the quadratic relations), or as sums of sixteen squares (for the biquadratic). The subject being extensive we shall not go into it here.

By the theorem of § 6, the Göpel relations for theta functions (and constants) are formally equivalent to the same for f-functions (and quasi-constants), so that the Kummer surface gives in this way 60 arithmetical duals. We shall sketch a means for translating the arithmetic into terms of configurations of lattice points in S_r (space of r dimensions), a lattice point being one all of whose coördinates are integers. Finally it will be shown that the lattice properties in S_r can be mapped onto a point configuration in S_3. In each case the configuration is formally equivalent to the arithmetical dual, which in turn is formally equivalent to the equation of the surface, so that the existence of the point configuration implies that of the surface, and conversely.

9. Let ν_a, ν_a' be the parameters (§ 3) of the f whose mark is a, and consider the tetrad $T \equiv (f_a, f_b, f_c, f_d)$. As the eight parameters ν_a, ν_a', $\cdots$, ν_d, ν_d' take all their possible values, we shall say that T generates a spread. The totality of all spreads obtained by choosing a, b, c, d and F (cf. § 3) in all possible ways is called f-space, and each T for constant values of the eight parameters involved is a point in f-space (or the coördinates of a point in f-space). The analogously defined $(\varphi_a, \varphi_b, \varphi_c, \varphi_d)$, in which the φ's are quasi-constants, is a singular point of f-space. The coördinates of a general point in f-space are (f_a, f_b, f_c, f_d), in which all the parameters are general. In the same way we define the general f-line $(f_a, f_b, f_c, f_d, f_e, f_g)$, $a, \cdots, g$ being any six marks, and similarly a singular f-line on replacing f's by φ's when all the marks are even. We remark that, the p's being products, (p_1, p_2, p_3, p_4) is clearly an f-point; and likewise for hexads of products and f-lines. The analogies between f-points and lines and the points and lines of S_3 are obvious, and need not be further elaborated.

Suppose now that between the coördinates $(f_1, f_2, f_3, f_4) \equiv (x, y, z, w)$ of an f-point there is a relation of the form (A), § 1. Remembering that by § 4 (end) quasi-constants are to be considered as being of degree zero, and that by § 5(3) each product in (A) is with respect to the same parameters n_1, n_2 which are general, we shall call (A) the equation of the surface E (§ 1), in f-space, $x = f_1$, $y = f_2$, $z = f_3$, $w = f_4$ the parametric equations of E, and n_1, n_2 the parameters of a point on E. Replacing f-functions throughout in the above by quasi-constants, we similarly define singular surfaces in f-space. The singular surfaces have no immediate geometrical

analogues; their interest is arithmetical, and need not be considered here.* In the same way is defined the line-equation of an f-surface or of a singular f-surface. As one parameter varies, the other being constant, the associated point on the surface traces out what may be called an f-curve; and the analogy with geometry may obviously be continued step by step. Thus, to define a node on the f-surface E, we equate to zero the four formal partial derivatives with respect to the f-coördinates. It does not follow, of course, that any one of the four new relations thus derived between products is true. But if it be possible in the four derived relations so to replace the variables (f-functions) x, y, z, w by quasi-constants $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ respectively that the relations become simultaneously true, $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ is defined to be a node on E. It follows from § 7 that the E of Kummer's S has sixteen nodes, and it is sufficiently evident that the geometry of nodes on S can be transferred to a 'geometry' of 'nodes' on E, and conversely.† All of this geometry on E has a simple interpretation in S_3 which we shall briefly consider next, showing how the translation is effected in a general case. It is not the most general case, for we have excluded the possibility of imaginary $[h]$, but the perfectly general case presents no difficulty, and is treated in essentially the same way. In the case considered we end with the identity of two point configurations in a lattice space of three dimensions; in the most general case the final result is that two pairs of point configurations are thus severally identical. The case treated appears to be the more important. It includes the E of Kummer's S.

10. Consider three fixed points U, V_1, V_2 and two concentric spheres σ_1, σ_2, centers at the origin of the respective radii a_1, a_2 in S_r (§ 9), and let T_j denote the tangent plane (tangent S_{r-1}) at the point P_j on σ_j $(j = 1, 2)$. Let δ_j, δ_{12}, δ_{21} denote the respective perpendicular distances from U to T_j, from P_1 to T_2 and from P_2 to T_1. It is easily seen that

$$a_1\delta_{21} + a_1^2 = a_2\delta_{12} + a_2^2.$$

With reference to rectangular axes in S_3 the point P_{12} whose coördinates

* Interpreted similarly to the non-singular surfaces in § 10 (end), they give point configurations lying in one plane.

† The equivalent arithmetic is obtained in an obvious manner. For example the the theorem that a particular set of six nodes is in one plane is formally equivalent to a theorem that six products (§ 5), in which $r = s = 2$, are linearly related. The six products are sums of terms of the form $[h]F(\xi_1, \xi_2, \xi_{12})$, where the ξ's refer in this example to all the representations of the parameters n_1, n_2 in two sums of four squares, either 2 or 4 of which are odd and the rest even. The number of odd squares, also the $[h]$, are not the same for all sets of six coplanar nodes. Finally, as in the equivalent theta theory, all these relations between (symbolic) products can be summed up in the statement that a certain matrix is orthogonal, cf. Hudson, loc. cit., chs. III, XVI.

are either of the identical triads

$$(a_1\delta_1 + a_1^2,\ a_2\delta_2 + a_2^2,\ a_1\delta_{21} + a_1^2), \qquad (a_1\delta_1 + a_1^2,\ a_2\delta_2 + a_2^2,\ a_2\delta_{12} + a_2^2),$$

is called the image through U of the pair P_1, P_2, or when U is understood, simply the image of P_1, P_2.

Let δ_j' denote the perpendicular distance of V_j to T_j. Then it is readily seen that if the squares of the radii a_1, a_2 are integers, and V_j, P_j lattice points, h_j, defined by

$$h_j = a_j\delta_j' + a_j^2, \qquad h \equiv h_1 + h_2, \qquad (j = 1, 2),$$

are integers. We shall call h_j the index of P_j, and h the index of P_{12}. *Henceforth P_1, P_2 are lattice points.* Images P_{12} are now segregated into four classes K_j $(j = 0, 1, 2, 3)$, all images in K_j having their indices $\equiv j \bmod 4$. We shall consider henceforth only K_0 and K_2. A configuration C_0 of images in S_3 is called even when all its images belong to K_0; if all the images of C_2 belong to K_2, C_2 is odd. Several even C_0's together are regarded as forming a single C_0; likewise for C_2's, so that the C_j is the logical sum of all the images in the several C_j', C_j'', $\cdots$,

$$C_j = C_j' + C_j'' + \cdots \quad (j = 1, 2).$$

If in C_j a particular image P_{12} occurs precisely k times, P_{12} is *multiple of order k* in C_j; and if all the images of C_0 coincide with all those of C_2, C_0 and C_2 are identical when and only when images occupying the same positions in both are of equal multiplicities.

Returning to § 5 we shall consider only the case in which h there defined is even, so that $[h]$ is real. In the notation of § 5 choose for the radii of σ_1, σ_2 of this section, $\sqrt{n_1}$, $\sqrt{n_2}$, let the origin be the common center, and take for U, V_j, P_j the points

$$U \equiv (0, 0, \cdots, 0, 1, 1, \cdots, 1), \quad (s \text{ zeros}, r - s \text{ units}),$$
$$V_j \equiv (h_{j1}, h_{j2}, \cdots, h_{jr}), \quad (j = 1, 2),$$
$$P_j \equiv (\nu_{j1}, \nu_{j2}, \cdots, \nu_{jr}), \quad (j = 1, 2).$$

Assume $r \geqq 4$ (the only cases of importance), so that, any integer > 0 being in several ways a sum of four integral squares, σ_j always passes through lattice points P_j. For the U, V_j, P_j as just defined, it is easily seen that P_{12}, the image through U of P_1, P_2, is, in the notation of § 5, (ξ_1, ξ_2, ξ_{12}), and that h as there defined is its index. Hence to each term $[h](\xi_1, \xi_2, \xi_{12})$ of the product in § 5 corresponds an image of odd or even index, and to all the terms in the product correspond an even and an odd configuration.

Suppose now that we have a homogeneous algebraic relation $H = 0$

of degree $(r - s)$ (cf. § 4 end) between k such products. The k even configurations corresponding respectively to the k products form a single C_0; likewise the k odd configurations form a single C_2, and by § 3 (end) we see at once that $H = 0$ is formally equivalent to the statement that these single odd and even configurations are identical.

It should be of interest, for physical reasons, to find the corresponding discrete image of the special case of Kummer's surface known as the wave surface. The arithmetic for this case appears to be less elegant than that for the general surface.

INCIDENCES OF STRAIGHT LINES AND PLANE ALGEBRAIC CURVES AND SURFACES GENERATED BY THEM.

By Arnold Emch.

1. Introduction.

According to Lüroth* the analytic method of investigating problems of this sort, as indicated by Clebsch,† on account of its complexity, is not practicable. Thus, even for the simple problems connected with incidences of straight lines and conics in ordinary space, Lüroth prefers to use a purely geometric method for their solution.

Some of these problems, involving merely the number of solutions in each case, have subsequently also been solved by the methods of enumerative geometry. But, without wishing to detract from the great value of this geometry, it must be said that the applications of its processes are purely mechanical and do not afford a clear insight into the geometric organism. For example, enumerative geometry does not teach how to get the solutions effectively, or constructively; it is satisfied with getting their number.

In this paper I shall show that the particular problems considered by Lüroth, and more general ones, can, after all, be solved by a relatively very simple analytical method.

2. Incidence of Point, Straight Line, Plane, and Surface.

Let $(a) \equiv (a_1, a_2, a_3, a_4)$ and $(b) \equiv (b_1, b_2, b_3, b_4)$, be two points determining a straight line l (hereafter we shall use simply the term *line*, meaning straight line). The homogeneous coördinates of this line are

$$\rho p_{ik} = a_i b_k - a_k b_i, \qquad i, k = 1, 2, 3, 4;\ i \neq k,$$

so that the p_{ik}'s satisfy the identity

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} \equiv 0.$$

The line l intersects a plane (α)

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = 0$$

* "Ueber die Anzahl der Kegelschnitte, welche acht Geraden im Raume schneiden," *Crelle's Journal*, Vol. 68, pp. 185–192 (1868).

† Ibid., Vol. 59, p. 1 ff.

in a point $(x) \equiv (x_1, x_2, x_3, x_4)$, whose coördinates are easily found as

$$(1)\qquad \begin{aligned} \rho x_1 &= \quad 0 \quad + \alpha_2 p_{12} + \alpha_3 p_{13} + \alpha_4 p_{14}, \\ \rho x_2 &= -\alpha_1 p_{12} + \quad 0 \quad + \alpha_3 p_{23} + \alpha_4 p_{24}, \\ \rho x_3 &= -\alpha_1 p_{13} - \alpha_2 p_{23} + \quad 0 \quad + \alpha_4 p_{34}, \\ \rho x_4 &= -\alpha_1 p_{14} - \alpha_2 p_{24} - \alpha_3 p_{34} + \quad 0 \quad . \end{aligned}$$

Incidentally, it is noticed that this represents a singular null-system, whose determinant is the square of the lefthand side of the identity satisfied by the coördinates of the line, and in which to every plane (α) not passing through l corresponds its point of intersection with l.

If we now substitute (1) in the point equation of a surface S_n of order n

$$(2)\qquad f(x_1, x_2, x_3, x_4) = 0,$$

we obtain an equation

$$(3)\qquad F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0,$$

which is homogeneous and of order n in the α's, and also in the p_{ik}'s, and of order 1 in the coefficients of (2). The values of (1) which satisfy (2) are evidently the coördinates of a point on the surface S_n, but, being in the form (1), they are evidently also the coördinates of a point on l and (α), and, consequently, the coördinates of a point of intersection of l and S_n. Any plane (α) which passes through a point of intersection of l and S_n, will therefore satisfy (3). Conversely, every solution (β) of (3) represents a plane through one of the points of intersection of l and S_n. This follows immediately from the identity

$$F(\beta_1, \beta_2, \beta_3, \beta_4) \equiv f(\beta_2 p_{12} + \beta_3 p_{13} + \beta_4 p_{14}, \cdots),$$

in which the point $\rho x_1 = 0 + \beta_2 p_{12} + \beta_3 p_{13} + \beta_4 p_{14}, \cdots$ evidently lies on S_n, (β) and l. From this follows that the lefthand side of (3) resolves into n linear factors in the α's, which, set equal to zero, represent the equations of the intersections of l and S_n, or the bundles of planes through these points. (3) is therefore a necessary and sufficient condition for the incidence of point, line, plane, and surface.

When a plane (α) passes through l then its coördinates expressed by the coördinates of l, and a parameter λ are

$$(4)\qquad \begin{aligned} \rho\alpha_1 &= -p_{34}, \\ \rho\alpha_2 &= \lambda p_{34}, \\ \rho\alpha_3 &= \lambda p_{42} + p_{14}, \\ \rho\alpha_4 &= \lambda p_{23} - p_{13}. \end{aligned}$$

For these values of the α's, the coördinates of (x) given by (1) vanish identically, i.e., the point (x) as the intersection of (α) and l is indetermined.

As (2) is of degree n, and as each x in (2) vanishes when the α's have the values (4), it is evident that for every set of values (4), determined by λ, (2) vanishes to the nth order; in other words, *every plane, determined by* λ *in (4), is an n-fold plane of the degenerate surface (3) of class n.* This is also geometrically evident. (3) consists of the n bundles of planes through the n points of intersection of l with S_n, and a plane through l is common to all n bundles, and is therefore n-fold in the totality of bundles as given by (3).

3. Application to Plane n-ics in Space Cutting a Fixed Independent Plane n-ic in n Points. The Surface E.

Considering general curves and surfaces, if not stated otherwise, a surface S_n of order n which passes through a fixed plane n-ic C_n^0 depends on

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n+3)}{2}$$

parameters. Let C_n be another plane n-ic cutting each of $[n(n+3)/2] - n$ lines l, and the C_n^0 in n collinear points. If S_n is also made to contain C_n, the number of independent parameters left for the complete determination of S_n is

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n+3)}{2} - \frac{n(n+3)}{2} + n = \frac{(n-1)n(n+1)}{6}.$$

Choosing this as the number of fixed points P through which S_n shall pass, imposes upon S_n (containing C_n^0)

$$\frac{n(n+3)}{2} + \frac{(n-1)n(n+1)}{6} = \frac{n(n^2+3n+8)}{6}$$

given conditions, so that there are

$$\frac{(n+1)(n+2)(n+3)}{6} - 1 - \frac{n(n^2+3n+8)}{6} = \frac{n(n+1)}{2}$$

independent parameters left. Hence there are $[n(n+1)/2]+1$ linearly independent surfaces ϕ_1, ϕ_2, ϕ_3, $\cdots$, $\phi_{[n(n+1)/2]+1}$ through C_n^0 and the $n(n^2-1)/6$ fixed points P, and the general S_n may be written in the form

$$(5) \qquad a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \cdots + a_{[n(n+1)/2]+1}\cdot\phi_{[n(n+1)/2]+1} = 0.$$

Every plane (α) cuts the C_n^0 and the $n(n+1)/2$ lines l in points of a plane n-ic C_n which together with C_n^0 and the fixed points P determine an S_n uniquely. But when (α) passes through one of the points P, the corresponding S_n degenerates into the plane (α) itself, and an $(n-1)$-ic.

Now let l_m be one of $[n(n+1)/2]+1$ independent lines l, and denote its

homogeneous coördinates by p_{ik}^{m}. Substituting these values in (1) gives the coördinates of the point of intersection (x) of the line l_m with the plane (α). Let $\phi_1^m, \phi_2^m, \phi_3^m, \cdots$ be the result of the substitution of the coördinates of (x) in $\phi_1, \phi_2, \phi_3, \cdots$; then

$$(6) \qquad a_1\phi_1^m + a_2\phi_2^m + a_3\phi_3^m + \cdots + a_{[n(n+1)/2]+1}\phi^m{}_{[n(n+1)/2]+1} = 0$$

is the condition that the point of intersection of l_m and (α) lies on S_n. Repeating this process for all lines l_m, $m = 1, 2, 3, \cdots, [n(n+1)/2]+1$; and the same (α), leads to $[n(n+1)/2]+1$ equations of order n in the α's, whose consistency is assured when the determinantal equation

$$(7) \qquad \begin{vmatrix} \phi_1^1 & \phi_2^1 & \phi_3^1 & \cdots & \phi^1_{[n(n+1)/2]+1} \\ \phi_1^2 & \phi_2^2 & \phi_3^2 & \cdots & \phi^2_{[n(n+1)/2]+1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \phi_1^{\,n(n+1)/2+1} & \phi_2^{\,n(n+1)/2]+1} & \phi_3^{[n(n+1)/2]+1} & \cdots & \phi^{[n(n+1)/2]+1}_{[n(n+1)/2]+1} \end{vmatrix} = 0$$

is satisfied. This is an equation of order

$$n\left(\frac{n(n+1)}{2}+1\right) = \frac{n^3+n^2+2n}{2}$$

in the α's. But the bundles of planes through the fixed points P, determining degenerate S_n's must be subtracted, so that the degree of the equation proper is

$$\frac{n^3+n^2+2n}{2} - \frac{n(n^2-1)}{6} = \frac{n(2n^2+3n+7)}{6}.$$

Every plane (α) of the reduced equation (7) cuts the C_n^0 in n points and the $[n(n+1)/2]+1$ lines l in points which lie on a plane n-ic C_n. Hence

THEOREM 1. *The planes, each of which cuts singly each of* $[n(n+1)/2]+1$ *lines* l *and an independent plane* n*-ic* C_n^0 *together in a system of points which lies on a plane* n*-ic* C_n*, form a surface* E *of class*

$$\frac{n(2n^2+3n+7)}{6}.$$

4. DEVELOPABLE SURFACE D WHOSE PLANES CUT A FIXED PLANE n-IC AND $[n(n+1)/2]+2$ INDEPENDENT LINES IN POINTS OF PLANE n-ICS.

Let E_1 be the surface determined by the lines

$$l_1, l_2, l_3, \cdots, l_{n(n+1)/2}, l_{[n(n+1)/2]+1};$$

E_2 the surface determined by the lines

$$l_1, l_2, l_3, \cdots, l_{n(n+1)/2}, l_{[n(n+1)/2]+2},$$

according to theorem (1), E_1 and E_2 have a developable surface D^* of class

$$\left[\frac{n(2n^2+3n+7)}{6}\right]^2$$

in common. But through each of the $n(n+1)/2$ lines common to both E_1 and E_2 there is a pencil of planes, whose planes cut the remaining lines and $C_n{}^0$ in points which lie on an n-ic for each such plane. These planes are each n-fold in E_1 and E_2, and as they all contain n-ics which cut one of the lines l improperly, each pencil of planes through the lines common to E_1 and E_2 must be deducted n^2-fold from D^*. This leaves for the class of the developable surface D proper the number

$$\left[\frac{n(2n^2+3n+7)}{6}\right]^2 - n^2\frac{n(n+1)}{2} = \frac{n^2(4n^4+12n^3+19n^2+24n+49)}{36}.$$

Hence

THEOREM 2. *The planes, each of which cuts singly each of* $[n(n+1)/2]+2$ *lines* l *and an independent plane n-ic* $C_n{}^0$ *together in a system of points which lies on a plane n-ic* C_n*, form a surface D of class*

$$\frac{n^2(4n^4+12n^3+19n^2+24n+49)}{36}.$$

5. NUMBER OF PLANE n-ICS CUTTING SINGLY EACH OF $[n(n+1)/2]+3$ LINES l AND A FIXED PLANE n-IC IN n POINTS.

The surface D determined by the lines

$$l_1, l_2, l_3, \cdots, l_{n(n+1)/2}, l_{[n(n+1)/2]+1}, l_{[n(n+1)/2]+2},$$

according to theorem (2), and the surface E determined by the lines

$$l_1, l_2, l_3, \cdots, l_{n(n+1)/2}, l_{[n(n+1)/2]+3},$$

according to theorem (1), have

$$\frac{n^2(4n^4+12n^3+19n^2+24n+49)}{36}\cdot\frac{n(2n^2+3n+7)}{6}$$

common solutions, including the improper ones. To get the number of the latter we must take into account that there are $n(2n^2+3n+7)/6$ planes through each of the $n(n+1)/2$ lines common to D and E, of which each cuts the remaining $[n(n+1)]/2+2$ lines of a plane n-ic belonging to both D and E. Each of these planes is a multiple plane of order n in both surfaces. The number of proper planes cutting singly all $[n(n+1)/2]+3$ lines in points on plane n-ics is therefore

$$\frac{n^2(4n^4+12n^3+19n^2+24n+49)}{36}\cdot\frac{n(2n^2+3n+7)}{6}$$

$$-\frac{n(n+1)}{2}\cdot\frac{n(2n^2+3n+7)}{6}\cdot n^2$$

$$=\frac{n^3(8n^6+36n^5+66n^4+99n^3+123n^2+89n+343)}{216}.$$

Denoting this number by N, the result may be stated in

THEOREM 3. *There are N plane n-ics which cut singly each of $[n(n+1)/2]+3$ lines and which cut, in each case, a fixed plane n-ic in n points.*

6. A CERTAIN SURFACE GENERATED BY PLANE n-ICS.

Given a fixed plane n-ic C_n^0 and $n(n+1)/2$ independent lines l. Consider the pencil of planes through another independent fixed line s. Every plane of this pencil cuts C_n^0 and the l's in points on an n-ic C_n. As the plane describes the pencil through s, the corresponding plane n-ic C_n describes a certain surface, whose order is determined by the number of points in which any other independent line, say $l_{[n(n+1)/2]+1}$ cuts the surface. Now we know the class of the surface E determined by $l_1, l_2, l_3, \cdots, l_{n(n+1)/2}, l_{[n(n+1)/2]+1}$. The number of plane n-ics through s cutting all these $[n(n+1)/2]+1$ lines l is, of course, equal to the class of E. Hence $l_{n(n+1)/2]+1}$ cuts the surface in a number of points equal to the class of E, and we have

THEOREM 4. *The plane n-ics, whose planes form a pencil, and which cut a fixed plane n-ic in n points, and which cut singly each of $n(n+1)/2$ lines, generate a surface of order*

$$\frac{n(2n^2+3n+7)}{6},$$

in which the axis s of the pencil is an

$$\left[\frac{n(2n^2+3n+7)}{6}-n\right]=\frac{n(2n^2+3n+1)}{6}$$

fold line.

The last part of this theorem is obvious, since a plane through s cuts the surface, outside of s, in a plane n-ic only.

7. EXAMPLE OF CIRCLES.

Let the surfaces S_n be spheres, so that the fixed plane n-ic C_n^0 is the imaginary sphero-circle (absolute) at infinity. In this case (as well as for any other conic as C_n^0) $[n(n+1)/2]+3=6$, $N=175$, and theorem 3 assumes the form of a corollary:

There are 175 circles cutting singly each of 6 independent lines.

8. GENERALIZATION OF LÜROTH'S PROBLEM.

In the paper referred to in the introduction Lüroth investigates incidences of conics and lines and certain surfaces enveloped by the planes of such conics, or generated by such conics. His concluding theorem states that there are 92 conics which cut singly each of 8 independent lines in space. I shall generalize this result by establishing the number of plane n-ics which cut singly each of $[n(n+3)/2]+3$ independent lines in space.

The key for the solution of this problem is again furnished by the establishment of the class of the surface E formed by the planes of the n-ics cutting singly each of $[n(n+3)/2]+1$ lines l. For this purpose I shall follow briefly a line of reasoning entirely similar to that for the surface E in § 3. Accordingly it is found that there are $(n^2+n+2)/2$ linearly independent surfaces $\phi_1, \phi_2, \phi_3, \cdots$ of order n which pass through n definite fixed lines l and $(n^3-3n^2+2n)/6$ independent fixed points P. Hence any surface S_n of order n through these fixed elements may be represented by a linear polynomial

$$S_n = a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + \cdots$$

in the ϕ's. The condition that a plane (α) shall cut $(n^2+n+2)/2$ other independent lines l in points which lie on an S_n is equivalent with the same number of equations of the form

$$a_1\phi_1{}^m + a_2\phi_2{}^m + a_3\phi_3{}^m + \cdots = 0.$$

This leads to a determinantal equation of order $n(n^2+n+2)/2$ in the α's. But, again, the bundles of planes through the points P must be deducted, so that a reduced equation of degree

$$\frac{n(n^2+n+2)}{2} - \frac{n^3-3n^2+2n}{6} = \frac{n^3+3n^2+2n}{6}$$

in the α's is left. Thus, in analogy with theorem (1) we have

THEOREM 5. *The planes each of which cuts singly each of* $[n(n+3)/2]+1$ *independent fixed lines in space in points which lie on a plane n-ic is a surface E of class*

$$\frac{n^3+3n^2+2n}{3}.$$

By Lüroth's method, extended to the case of n-ics, and in analogy with the procedure in §§ 4 and 5, it is found that the class of the developable surface D, formed by the planes which cut $[n(n+3)/2]+2$ lines l in points on n-ics, is

$$\frac{n^2(2n^4+12n^3+17n^2-3n+8)}{18}.$$

In case of conics, $n = 2$, this number, as shown by Lüroth, has to be reduced by 10, so that for conics the class of D is 34.

Finally the number of plane n-ics cutting $[n(n + 3)/2] + 3$ independent lines in space is

$$\frac{n^3(n^2 + 3n + 2)(n^4 + 6n^3 + 4n^2 - 15n + 4)}{27}.$$

For $n = 2$ this number is reduced by 20 (Lüroth, loc. cit.) so that the number of conics cutting singly each of 8 lines is 92.

ON THE THEOREMS OF GAUSS AND GREEN.

By Vincent C. Poor.

Part I.

The Divergence of a Vector.

Definitions.—Among the definitions for the divergence of a vector, we have the following:

$$\operatorname{div} \mathbf{u} = \operatorname*{Limit}_{\tau \to 0} \frac{\int \mathbf{u} \times \mathbf{n}\, d\sigma}{\tau},$$

where $\mathbf{u}$ is a continuous vector point function whose scalar product $\mathbf{u} \times \mathbf{n}$ with the outward unit normal $\mathbf{n}$ of the surface σ is integrable over the boundary of the region τ, or over the bounding surface of any sub-region of the region τ. This definition may be looked upon as a differential form of Gauss's Theorem:

$$\int \operatorname{div} \mathbf{u}\, d\tau = \int \mathbf{u} \times \mathbf{n}\, d\sigma. \tag{1}$$

In the elements of vector analysis, Burali-Forti and Marcolongo give the following definition:

$$\operatorname{div} \mathbf{u} = \{\operatorname{grad} (\mathbf{u} \times \mathbf{a}) - \operatorname{rot} (\mathbf{u} \wedge \mathbf{a})\} \times \mathbf{a},^*$$

where $\mathbf{a}$ is an arbitrary constant vector. The symbol, $\wedge$, read vec, is Burali-Forti and Marcolongo's symbol for the vector product of two vectors, and the abbreviations, grad and rot, are the gradient, and the rotation or curl of a vector respectively.

The former definition is not convenient for proving directly such theorems as

$$\begin{aligned} \operatorname{div} m\mathbf{u} &= m \operatorname{div} \mathbf{u} + \operatorname{grad} m \times \mathbf{u}, \\ \operatorname{div} \mathbf{u} \wedge \mathbf{v} &= \mathbf{u} \times \operatorname{rot} \mathbf{v} - \mathbf{v} \times \operatorname{rot} \mathbf{u} \end{aligned}$$

and the uniqueness and existence of the limit might be difficult to establish without resorting to a coördinate system. The second definition seems very artificial, to say the least, and not of a form to be readily remembered.

Some of these objections are obviated through the following definition, which, expressed in the notation of Burali-Forti and Marcolongo, reads as follows:

$$\operatorname{div} \mathbf{u} \cdot dP \times \delta P \wedge \vartheta P = d\mathbf{u} \times \delta P \wedge \vartheta P + \delta \mathbf{u} \times \vartheta P \wedge dP + \vartheta \mathbf{u} \times dP \wedge \delta P, \tag{2}$$

where the point differentials dP, δP, ϑP are any three non-complanar vec-

* "Elements de Calcul Vectoriel, Burali-Forti et Marcolongo," p. 71, article 3.

tors. This form is both natural and convenient; natural, since it may be easily seen to be a differential form of Gauss' Theorem, and convenient because of the differential operators involved. The expanded forms of div $m\mathbf{u}$ and div $\mathbf{u} \wedge \mathbf{v}$ are readily deducible. By certain changes this form of definition may be deduced from Burali-Forti's definition* for the first invariant of a particular homography.

Gauss' Theorem.—Also from this definition the integral form of Gauss' Theorem may be established. To show this, the region τ, bounded by the surface σ, may be divided into cells by passing planes parallel to the planes determined by the arbitrary vectors dP, δP, ϑP, taken in pairs. The right member of (2) is the resultant flux of the vector $\mathbf{u}$, through the surface of each cell. The resultant flux of $\mathbf{u}$ through each face common to two adjacent cells is zero, since the outward normals are of opposite sense. Upon applying the fundamental law of the integral calculus the right member in the limit is seen to be the flux integral of the vector $\mathbf{u}$ through the surface σ, and Gauss' Theorem follows.

Uniqueness.—To prove the uniqueness of the idea defined by (2) we can, in the usual way, assume another operator operating on $\mathbf{u}$, div′ $\mathbf{u}$. Then by definition

$$(3)\quad \text{div}'\,\mathbf{u}\cdot dP \times \delta P \wedge \vartheta P = d\mathbf{u} \times \delta P \wedge \vartheta P + \delta\mathbf{u} \times \vartheta P \wedge dP + \vartheta\mathbf{u} \times dP \wedge \delta P.$$

Subtracting (3) from (2) we have

$$(\text{div}\,\mathbf{u} - \text{div}'\,\mathbf{u}) dP \times \delta P \wedge \vartheta P \equiv 0.$$

But since the volume element $dP \times \delta P \wedge \vartheta P$ is arbitrary,

$$\text{div}\,\mathbf{u} = \text{div}'\,\mathbf{u}.$$

Existence.—The existence of div $\mathbf{u}$ depends on the existence of $d\mathbf{u}$, which in turn exists if

$$\operatorname*{Limit}_{h \to 0} \frac{\mathbf{u}(P + hdP) - \mathbf{u}(P)}{h}$$

exists.

Or, if $\mathbf{u}$ is a function of the distance r, we may write

$$\begin{aligned}
\text{div}\,\mathbf{u}\, d\tau &= \frac{\partial \mathbf{u}}{\partial r} dr \times \delta P \wedge \vartheta P + \frac{\partial \mathbf{u}}{\partial r} \delta r \times \vartheta P \wedge dP + \cdots \\
&= \frac{\partial \mathbf{u}}{\partial r} (\text{grad}\, r \times dP) \times \delta P \wedge \vartheta P + \cdots + \cdots \\
&= \text{grad}\, r \times \left[\frac{\partial \mathbf{u}}{\partial r} \times \delta P \wedge \vartheta P \cdot dP + \frac{\partial \mathbf{u}}{\partial r} \times \vartheta P \wedge dP \cdot \delta P \right. \\
&\qquad \left. + \frac{\partial \mathbf{u}}{\partial r} \times dP \wedge \delta P \cdot \vartheta P \right] \\
&= \text{grad}\, r \times \frac{\partial \mathbf{u}}{\partial r} \cdot dP \times \delta P \wedge \vartheta P,
\end{aligned}$$

* "Transformations Linéaires, Burali-Forti et Marcolongo," p. 23, Article 3.

since the vector $dP \times \delta P \wedge \vartheta P \cdot (\partial \mathbf{u}/\partial r)$ is expressible linearly in terms of the vectors dP, δP, and ϑP, in the form given in square brackets. Hence,

$$\operatorname{div} \mathbf{u} = \operatorname{grad} r \times \frac{\partial \mathbf{u}}{\partial r}.$$

Thus div $\mathbf{u}$ is seen to exist if $\partial \mathbf{u}/\partial r$ exists. Or, in rectangular coördinates, we may say that the divergence of the vector $\mathbf{u}$ exists if the partial derivatives of $\mathbf{u}$, with respect to x, y, and z, exist.

Example.—Let us apply the definition (1) to the vector point function $\mathbf{r} = P - O$, O being a fixed point. We thus have, since $d\mathbf{r} = dP$,

$$\operatorname{div} \mathbf{r} \cdot d\tau = dP \times \delta P \wedge \vartheta P + \delta P \times \vartheta P \wedge dP + \vartheta P \times dP \wedge \delta P.$$

Since each of the scalar triple products in the right member is the same and equal to $d\tau$, this right member becomes $3d\tau$, so that div $\mathbf{r} = 3$.

PART II.

Green's Theorem.

A General Form.—Turning from the elements to the more advanced part of vector analysis, we find in the "Transformations Linéaires" of Burali-Forti and Marcolongo the divergence of a vector defined as the first invariant of the homography* $d\mathbf{u}/dP$, written

$$\operatorname{div} \mathbf{u} \equiv I_1 \left(\frac{d\mathbf{u}}{dP} \right).$$

From this definition for div $\mathbf{u}$, the following theorem may be obtained:

$$\operatorname{div} \alpha u = I_1 \left(\alpha \frac{d\mathbf{u}}{dP} \right) + \operatorname{grad} K\alpha \times \mathbf{u}, \tag{4}$$

where $K\alpha$ is the conjugate of the homography α. We are now able to demonstrate the following very general form of Green's Theorem:

$$\int I_1 \left(K\alpha \cdot \frac{d \operatorname{grad} \beta}{dP} - K\beta \frac{d \operatorname{grad} \alpha}{dP} \right) d\tau = \int (K\alpha \operatorname{grad} \beta - K\beta \operatorname{grad} \alpha) \times \mathbf{n}\, d\sigma, \tag{5}$$

where α and β are homographies.

* A homography is defined as any linear operator which transforms a vector into a vector.

In rectangular coördinates, the homography

$$\alpha \equiv \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

operating on the vector

$$\mathbf{u} = u_1 i + u_2 j + u_3 k$$

changes the vector u into the vector

$$\alpha \mathbf{u} = (a_{11}u_1 + a_{12}u_2 + a_{13}u_3)\mathbf{i} + (a_{21}u_1 + a_{22}u_2 + a_{23}u_3)\mathbf{j} + (a_{31}u_1 + a_{32}u_2 + a_{33}u_3)\mathbf{k}.$$

Proof.—If we substitute (4) in Gauss' Theorem (1), we will have

$$(6) \qquad \int I_1\left(\alpha\frac{d\mathbf{u}}{dP}\right)d\tau - \int \alpha\mathbf{u}\times\mathbf{n}d\sigma = -\int \text{grad } K\alpha\times\mathbf{u}\,d\tau.$$

If we now take $\mathbf{u} = \text{grad } K\beta$, (6) becomes

$$(7) \qquad \int I_1\left(\alpha\frac{d\ \text{grad } K\beta}{dP}\right)d\tau - \int \alpha\ \text{grad } K\beta\times\mathbf{n}\,d\sigma = -\int \text{grad } K\alpha\times\text{grad } K\beta\,d\tau,$$

a form of Green's Theorem. Since the right member of (7) is symmetric in α and β, the left member is likewise, so that α and β may be interchanged without affecting the truth of equation (7). Thus:

$$(8) \qquad \int I_1\left(\beta\frac{d\ \text{grad } \alpha}{dP}\right)d\tau - \int \beta\ \text{grad } K\alpha\times\mathbf{n}\,d\sigma = -\int \text{grad } K\alpha\times\text{grad } K\beta\,d\tau.$$

Subtracting (8) from (7) and transposing, we have

$$(9) \qquad \int I_1\left(\alpha\frac{d\ \text{grad } K\beta}{dP} - \beta\frac{d\ \text{grad } K\alpha}{dP}\right)d\tau = \int(\alpha\ \text{grad } K\beta - \beta\ \text{grad } K\alpha)\times\mathbf{n}\,d\sigma.$$

Equation (5) is true for any two homographies. It is therefore true if we replace α and β by $K\alpha$ and $K\beta$. If we make this substitution in (9) and remember that $KK\alpha = \alpha$ we obtain equation (5).

Special Forms.—The operators I_1 and K are linear operators, changing a homography into a homography. Thus

$$I_1(m\alpha) = mI_1(\alpha),$$

if m is a scalar point function. Also $K(m) = m$. If, then, we replace the homographies α and β by the scalar point functions φ and ψ, equation (5) reduces to

$$\int\left[\varphi I_1\left(\frac{d\ \text{grad}\,\psi}{dP}\right) - \psi I_1\left(\frac{d\ \text{grad}\,\varphi}{dP}\right)\right]d\tau = \int(\varphi\ \text{grad}\,\psi - \psi\ \text{grad}\,\varphi)\times\mathbf{n}\,d\sigma.$$

If we denote div grad by Δ this last equation may be written

$$\int(\varphi\Delta\psi - \psi\Delta\varphi)d\tau = \int(\varphi\ \text{grad}\,\psi - \psi\ \text{grad}\,\varphi)\times\mathbf{n}\,d\sigma,$$

the well-known form of Green's Theorem, which appears as a special instance of the more general equation (5). Further if α is taken as unity and grad β as the vector $\mathbf{u}$, (5) reduces to Gauss' Theorem as it should.

Another theorem, closely allied to these, but which may better be classed with the theorems in a previous paper* by the writer, will be given without proof. The proof may follow the lines suggested by that paper.

THEOREM.—*If* α *is a homography symmetrical in* P *and* M *such that* $d\alpha/dM = -(d\alpha/dP)$ *and if* $\mathbf{u}$ *is independent of* P, *the point of integration, then:*

$$\int \left(\frac{d\alpha}{dP}\mathbf{u}\right)\mathbf{x}\,d\tau = -\int \mathbf{u}\times\mathbf{n}\,\alpha\mathbf{x}\,d\sigma - \int \alpha\frac{d\mathbf{x}}{dP}\mathbf{u}\,d\tau.$$

* *Bulletin American Mathematical Society*, Vol. 22, Jan., 1916, p. 174. "Transformations in the Theory of the Linear Vector Function."

AN EXTENSION OF THE STURM-LIOUVILLE EXPANSION.

By Chester Claremont Camp.

Introduction.

In 1836–7 C. Sturm's formal development of a more or less arbitrary function $f(x)$ in terms of solutions of the self-adjoint equation

$$\frac{d}{dx}\left(k\frac{du}{dx}\right) + (\lambda g - l)u = 0$$

and the Sturmian boundary conditions

$$\left.\begin{aligned} \alpha u(a) + \alpha' u'(a) &= 0, \\ \beta u(b) + \beta' u'(b) &= 0, \end{aligned}\right\} \qquad \left.\begin{aligned} |\alpha| + |\alpha'| &\neq 0 \\ |\beta| + |\beta'| &\neq 0 \end{aligned}\right\}$$

was considered by J. Liouville,* who undertook the problem of showing that the series converges and that its value is $f(x)$. His work is important although it did not satisfy all the requirements of modern mathematical rigor, but in two remarkable papers A. Kneser† completely settled all the more fundamental questions concerning the development. It remained for Haar‡ several years later to give the solution finality.

In 1908 Birkhoff || extended the theory not only to equations of the nth order of the form

$$\frac{d^n u}{dx^n} + p_1\frac{d^{n-1}u}{dx^{n-1}} + \cdots + p_{n-1}\frac{du}{dx} + (p_n + \lambda g)u = 0,$$

but to systems no longer self-adjoint and conditions no longer Sturmian.

The theory is capable of extension in several directions. Bôcher§ considered a system of two homogeneous linear differential equations of the first order and studied the roots of a solution without regard to boundary conditions. Schlesinger¶ took a system of n linear equations of the first order with coefficients which are rational in x and obtained the asymptotic forms for a solution.

The object of this paper is to discuss an extension of a problem recently considered by Professor Hurwitz, namely the simultaneous expansion of two

* *Liouville's Journal*, Vol. 1 (1836), p. 253; Vol. 2 (1837), p. 16 and p. 418.

† *Math. Ann.*, Vol. 58, p. 81 and Vol. 60, p. 402.

‡ Goettingen dissertation (1909) reprinted in *Math. Ann.*, Vol. 69 (1910), p. 331. Also a second paper, *Math. Ann.*, Vol. 71 (1911), p. 38. See also Mercer, *Phil. Trans.*, Vol. 211 (1911), p. 111.

|| *Trans. Amer. Math. Soc.*, Vol. 9, p. 373.

§ *Trans. Amer. Math. Soc.*, Vol. 3 (1902), p. 196.

¶ *Math. Ann.*, Vol. 63 (1907), p. 277.

functions $f(x)$, $g(x)$ in terms of solutions of the system

$$\frac{du}{dx} = [\lambda + a(x)]v,$$

$$\frac{dv}{dx} = -[\lambda + b(x)]u,$$

and the boundary conditions

$$\alpha u(a) + \beta v(a) = 0,$$
$$\alpha' u(b) + \beta' v(b) = 0.$$

Although this system is simple, it is not self-adjoint and the functions $a(x)$, $b(x)$ are any two real functions which possess continuous second derivatives.

The extension I have made is that resulting from the substitution of the more general linearly independent boundary conditions

$$\alpha_1 u(a) + \beta_1 v(a) + \gamma_1 u(b) + \delta_1 v(b) = 0,$$
$$\alpha_2 u(a) + \beta_2 v(a) + \gamma_2 u(b) + \delta_2 v(b) = 0,$$

in which the coefficients are such as to satisfy

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{vmatrix} \neq 0.$$

I wish to acknowledge my indebtedness to Professor Hurwitz for his constant interest and careful direction as well as for his recent article as a basis for research.

For further references the reader is pointed to the following by Bôcher: Encyklopædie der Math. Wiss., Band II, Heft 4, pp. 437–463; Proceedings of the Fifth International Congress of Mathematicians, Vol. I (1912), pp. 163–195.

Section I. Preliminary Lemmas.

Consider the system

$$\begin{cases} N_1(uv\lambda) \equiv u'(x) - [\lambda + a(x)]v(x) = 0, \\ N_2(uv\lambda) \equiv v'(x) + [\lambda + b(x)]u(x) = 0, \end{cases} \tag{1}$$

with the boundary conditions

$$\begin{cases} U_1(uv) \equiv \alpha_1 u(0) + \beta_1 v(0) + \gamma_1 u(1) + \delta_1 v(1) = 0, \\ U_2(uv) \equiv \alpha_2 u(0) + \beta_2 v(0) + \gamma_2 u(1) + \delta_2 v(1) = 0, \end{cases} \tag{2}$$

in which λ is real or complex; $a(x)$, $b(x)$ are real functions possessing con-

tinuous second derivatives for $0 \leqq x \leqq 1$; and the coefficients of u, v, in (2) are real constants such that the matrix*

$$\left\| \begin{matrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{matrix} \right\|$$

is of rank two and

$$(\alpha\beta) = (\gamma\delta) \neq 0 \tag{3}$$

where

$$(\alpha\beta) \equiv \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}.$$

The solution $u(x) \equiv v(x) \equiv 0$ is called the *trivial* solution of (1), (2). All other solutions are termed *non-trivial.*

Studying non-trivial solutions of (1), (2) we derive the following lemmas.

LEMMA I: *If* (u_m, v_m) *and* (u_n, v_n) *are two solutions of*

$$N_1(u_k v_k \lambda_k) = N_2(u_k v_k \lambda_k) = 0,$$

then

$$\left[u_m(x)v_n(x) - u_n(x)v_m(x)\right]_{x=0}^{x} = (\lambda_m - \lambda_n)\int_0^x \left[u_m(s)u_n(s) + v_m(s)v_n(s)\right]ds. \tag{4}$$

By hypothesis

$$\begin{aligned} u_m'(x) - \left[\lambda_m + a(x)\right]v_m(x) &= 0, \\ v_m'(x) + \left[\lambda_m + b(x)\right]u_m(x) &= 0, \\ u_n'(x) - \left[\lambda_n + a(x)\right]v_n(x) &= 0, \\ v_n'(x) + \left[\lambda_n + b(x)\right]u_n(x) &= 0. \end{aligned}$$

If we multiply these equations by $v_n(x)$, $-u_n(x)$, $-v_m(x)$, and $u_m(x)$ respectively, add, and integrate from 0 to x, we obtain the equation (4) required.

LEMMA II: *If* (u, v) *satisfies* $N_1 = \varphi$, $N_2 = \psi$, *where* φ, ψ *are functions of* x *continuous* $0 \leqq x \leqq 1$, *and* (U, V) *satisfies* $N_1 = N_2 = 0$ *for the same value of* λ, *then*

$$\left[u(x)V(x) - U(x)v(x)\right]_0^x = \int_0^x \left[V(s)\varphi(s) - U(s)\psi(s)\right]ds. \tag{5}$$

By hypothesis

$$\begin{aligned} u'(x) - \left[\lambda + a(x)\right]v(x) &= \varphi(x), \\ v'(x) + \left[\lambda + b(x)\right]u(x) &= \psi(x), \\ U'(x) - \left[\lambda + a(x)\right]V(x) &= 0, \\ V'(x) + \left[\lambda + b(x)\right]U(x) &= 0. \end{aligned}$$

* If the matrix is of rank less than two the conditions (2) are not independent. The condition $(\alpha\beta) = (\gamma\delta)$ is somewhat analogous to the condition of self-adjointness for single equations of order two. The case in which $(\alpha\beta) = (\gamma\delta) = 0$ can obviously be reduced to the problem considered by Professor W. A. Hurwitz.

Multiplying respectively by $V(x)$, $-U(x)$, $-v(x)$, $u(x)$; adding and integrating as before we get equation (5) above.

LEMMA III: *Two pairs of functions* $u(x)$, $v(x)$; $U(x)$, $V(x)$; *satisfying* $U_1 = U_2 = 0$, *satisfy also the relation*

$$[u(x)V(x) - v(x)U(x)]_0^1 = 0 \tag{6}$$

when (3) *is satisfied.*

We are given

$$\begin{aligned}
\alpha_1 u(0) + \beta_1 v(0) + \gamma_1 u(1) + \delta_1 v(1) &= 0,\\
\alpha_2 u(0) + \beta_2 v(0) + \gamma_2 u(1) + \delta_2 v(1) &= 0,\\
\alpha_1 U(0) + \beta_1 V(0) + \gamma_1 U(1) + \delta_1 V(1) &= 0,\\
\alpha_2 U(0) + \beta_2 V(0) + \gamma_2 U(1) + \delta_2 V(1) &= 0.
\end{aligned}$$

Since $(\alpha\beta) \neq 0$,

$$u(0) = -\frac{1}{(\alpha\beta)}\begin{vmatrix} \gamma_1 u(1) + \delta_1 v(1), & \beta_1 \\ \gamma_2 u(1) + \delta_2 v(1), & \beta_2 \end{vmatrix},$$

and

$$v(0) = -\frac{1}{(\alpha\beta)}\begin{vmatrix} \alpha_1, & \gamma_1 u(1) + \delta_1 v(1) \\ \alpha_2, & \gamma_2 u(1) + \delta_2 v(1) \end{vmatrix},$$

also $U(0)$, $V(0)$ are similar functions of $U(1)$, $V(1)$.

Substituting these values in (6) we get for the condition necessary that it be satisfied

$$(\alpha\beta)^2 = (\delta\beta)(\gamma\alpha) + (\gamma\beta)(\alpha\delta) \equiv (\alpha\beta)(\gamma\delta)$$

which is evidently true by (3).

LEMMA IV: *If* (u_1, v_1), (u_2, v_2) *are two linearly independent pairs of functions of* x, *continuous* $0 \leqq x \leqq 1$, *then*

$$G_2 \equiv \begin{vmatrix} (11) & (12) \\ (21) & (22) \end{vmatrix}$$

is real and greater than zero, where

$$(rs) \equiv \int_0^1 [u_r(s)\bar{u}_s(s) + v_r(s)\bar{v}_s(s)]ds$$

and $\bar{u}(x)$ *represents the conjugate of* $u(x)$.

Obviously $\bar{G}_2 = G_2$, i.e., G_2 is real.

Let

$$p(x) \equiv \begin{vmatrix} (11) & u_1(x) \\ (21) & u_2(x) \end{vmatrix},$$

$$q(x) \equiv \begin{vmatrix} (11) & v_1(x) \\ (21) & v_2(x) \end{vmatrix},$$

i.e.,

$$\begin{aligned}
p(x) &\equiv c_1 u_1(x) + c_2 u_2(x),\\
q(x) &\equiv c_1 v_1(x) + c_2 v_2(x).
\end{aligned}$$

Calculate

$$\int_0^1 [p(x)\bar{u}_k(x) + q(x)\bar{v}_k(x)]dx \equiv \begin{vmatrix} (11) & (1k) \\ (21) & (2k) \end{vmatrix} = \begin{cases} 0, \text{ if } k = 1, \\ G_2, \text{ if } k = 2. \end{cases}$$

Then

$$\begin{aligned} \int_0^1 (|p(x)|^2 + |q(x)|^2)dx &\equiv \int_0^1 [p(x)\bar{p}(x) + q(x)\bar{q}(x)]dx \\ &\equiv \bar{c}_1 \int_0^1 [p(x)\bar{u}_1(x) + q(x)\bar{v}_1(x)]dx + \bar{c}_2 \int_0^1 [p(x)\bar{u}_2(x) + q(x)\bar{v}_2(x)]dx \\ &\equiv \bar{c}_2 G_2 \geqq 0. \end{aligned}$$

If $G_2 = 0$ and $\bar{c}_2 = c_2 \neq 0$, then $p(x) \equiv q(x) \equiv 0$ and (u_1, v_1), (u_2, v_2) will be linearly dependent.

If $G_2 = 0$ and $\bar{c}_2 = 0$, then since

$$c_2 \equiv (11) \equiv \int_0^1 (|u_1(x)|^2 + |v_1(x)|^2)dx = 0,$$

$u_1 \equiv v_1 \equiv 0$ and they will again be linearly dependent, thus violating the hypothesis.

Hence

$$G_2 > 0 \tag{7}$$

since $\bar{c}_2 > 0$, and the lemma is proved.*

LEMMA V: *If* (3) *is satisfied, then*

$$[(\gamma\delta) + (\alpha\beta)]^2 \leqq [(\alpha\gamma) + (\beta\delta)]^2 + [(\alpha\delta) + (\gamma\beta)]^2. \tag{8}$$

(8) can easily be shown to be equivalent to

$$(\gamma\delta)^2 + (\alpha\beta)^2 \leqq (\alpha\gamma)^2 + (\beta\delta)^2 + (\alpha\delta)^2 + (\gamma\beta)^2. \tag{9}$$

But

$$(\alpha\gamma)^2 + (\beta\delta)^2 \geqq 2(\alpha\gamma)(\beta\delta),$$

and

$$(\alpha\delta)^2 + (\beta\gamma)^2 \geqq -2(\alpha\delta)(\beta\gamma).$$

Hence the right member of (9) is

$$\begin{aligned} &\geqq 2(\alpha\gamma)(\beta\delta) - (\alpha\delta)(\beta\gamma) \\ &\geqq 2(\alpha\beta)(\gamma\delta), \text{ or by (3)} \\ &\geqq (\alpha\beta)^2 + (\gamma\delta)^2. \end{aligned}$$

Since (9) is true, (8) is also.

LEMMA VI: *If* (3) *holds, it is impossible for the coefficients in* (2) *to satisfy*

$$\begin{aligned} (\alpha\gamma) + (\beta\delta) &= 0, \\ (\alpha\delta) + (\gamma\beta) &= 0. \end{aligned} \tag{10}$$

---

* The lemma admits of the following generalization: If (u_1, v_1), (u_2, v_2), $\cdots$ (u_n, v_n) are n linearly independent pairs of functions of x, continuous $0 \leqq x \leqq 1$, then

$$G_n \equiv \begin{vmatrix} (11) & (12) & \cdots & (1n) \\ (21) & (22) & \cdots & (2n) \\ \cdot & \cdot & \cdot & \cdot \\ (n1) & (n2) & \cdots & (nn) \end{vmatrix}$$

is real and greater than zero, n being a positive integer.

By (3)

$$(\alpha\beta) - (\gamma\delta) = 0.$$

If then we assume (10) true, by squaring and adding all these equations, we have

$$(\alpha\gamma)^2 + (\beta\delta)^2 + (\alpha\delta)^2 + (\gamma\beta)^2 + (\alpha\beta)^2 + (\gamma\delta)^2 + 2(\alpha\gamma)(\beta\delta) + 2(\alpha\delta)(\gamma\beta) - 2(\alpha\beta)(\gamma\delta) = 0.$$

The sum of the last three terms of the first member vanishes identically so that if (10) holds, each of the six determinants must vanish. Since this violates (3) the proof is complete.

Section II. Properties of Solutions of the Homogeneous and Non-homogeneous Systems.

LEMMA I: *A necessary and sufficient condition for the existence of a solution* (u, v) *of* (1), (2) *is that the determinant*

$$D(\lambda) \equiv \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_1v_1) & U_2(u_2v_2) \end{vmatrix} \tag{11}$$

vanish for some value of λ, *where* (u_1, v_1), (u_2, v_2) *are solutions of* (1) *defined by*

$$\begin{cases} u_1(0) = 1, & v_1(0) = 0, \\ u_2(0) = 0, & v_2(0) = 1. \end{cases} \tag{12}$$

By the existence theorem we know that a solution either of (1) or of the corresponding non-homogeneous system

$$N_1 = \varphi, \qquad N_2 = \psi,$$

where φ, ψ are functions of x, continuous $0 \leqq x \leqq 1$, will be entire in λ, provided it is defined by

$$\begin{cases} u(0) = c_1, \\ v(0) = c_2, \end{cases}$$

in which the c's are independent of λ. Clearly $D(\lambda)$ will also be entire in λ.

Any solution of (1) is expressible in the form

$$\begin{cases} u(x) = u(0)u_1(x) + v(0)u_2(x), \\ v(x) = u(0)\,v_1(x) + v(0)\,v_2(x). \end{cases} \tag{13}$$

In order that $u(x)$, $v(x)$ satisfy (2) it is necessary and sufficient that $u(0)$, $v(0)$ be determined so as to satisfy

$$\begin{cases} U_1(u) \equiv U_1(u_1v_1)u(0) + U_1(u_2v_2)v(0) = 0, \\ U_2(u) \equiv U_2(u_1v_1)u(0) + U_2(u_2v_2)v(0) = 0, \end{cases} \tag{14}$$

or if $u(x)$, $v(x)$ is to be a non-trivial solution, the determinant of coefficients of $u(0)$, $v(0)$, which is $D(\lambda)$, must vanish. This is obviously also sufficient.

LEMMA II: *The roots of* $D(\lambda)$ *are all real and the solutions of* (1), (2) *may be taken as real.*

Let $\lambda = \lambda_k$ be a root of $D(\lambda)$. Then (u_k, v_k) satisfies $N_1(u_k v_k \lambda_k) = 0$, $N_2(u_k v_k \lambda_k) = 0$ and $(\bar{u}_k, \bar{v}_k)$ will satisfy the same conditions for $\lambda = \bar{\lambda}_k$. Hence by Lemma I, Section I

$$[\bar{u}_k v_k - u_k \bar{v}_k]_0^x = (\bar{\lambda}_k - \lambda_k) \int_0^x (|u_k(s)|^2 + |v_k(s)|^2) ds.$$

Moreover $(\bar{u}_k, \bar{v}_k)$ satisfies $U_1(u) = U_2(u) = 0$ so that by Lemma III, Section I

$$(\bar{\lambda}_k - \lambda_k) \int_0^1 (|u_k|^2 + |v_k|^2) dx = 0.$$

If $\bar{\lambda}_k \neq \lambda_k$, (u_k, v_k) is a trivial solution which we have excluded. Therefore, $\bar{\lambda}_k = \lambda_k$ and λ_k is real.

Since each equation of the system (1), (2) is linear and $a(x)$, $b(x)$ are real, any solution of the system if complex may be broken up into real and imaginary parts which will separately satisfy the same system. When $D(\lambda)$ vanishes the ratio of $u(0)$ to $v(0)$ or of $v(0)$ to $u(0)$ is determinable* and by the existence theorem the solution $u(x)$, $v(x)$ is then uniquely determined except for a constant factor. Hence the coefficients of the real and imaginary parts constitute two solutions which are linearly dependent. It is therefore clear that we may restrict ourselves to real solutions of the system.

A similar discussion will show that we need consider only real solutions of the system

$$N_1 = \varphi, \qquad N_2 = \psi, \tag{15}$$

where φ, ψ are real functions of x and λ is restricted to real values, a restriction which we shall henceforth make.

A value of λ which makes $D(\lambda)$ vanish is called a *principal parameter value* and the corresponding solution of the homogeneous system (1), (2) is known as a principal solution.

Since $D(\lambda)$ is an entire function it cannot have more than a finite number of roots in any finite interval of the λ-axis, so that we may designate the roots of $D(\lambda)$ by λ_n, $n = 0, \pm 1, \pm 2, \cdots$.

THEOREM I: *A sufficient condition for the existence of a solution* $(\tilde{u}, \tilde{v})$ *of* (15), (2) *analytic for all real finite values of* λ *is that*

$$\int_0^1 \begin{vmatrix} u_n(s) & \varphi(s) \\ v_n(s) & \psi(s) \end{vmatrix} ds = 0 \tag{16}$$

for every (u_n, v_n) *satisfying* (1), (2) *for* $\lambda = \lambda_n$, $n = 0, \pm 1, \pm 2, \cdots$.

We shall consider in turn three kinds of values of λ:

Case I: Those for which the matrix of (11) is of rank two;

* If the matrix of (11) is of rank zero (Case III below), then this is found by the evaluation of an indeterminate form.

Case II: Those for which the matrix is of rank one (at $\lambda = \lambda_n$);
Case III: Those for which it is of rank zero (at $\lambda = \lambda_k$).

Take a solution of (1),

$$\begin{cases} u(x) = c_1 u_1(x) + c_2 u_2(x), \\ v(x) = c_1 v_1(x) + c_2 v_2(x), \end{cases} \tag{17}$$

which is identical with (13), provided $u(0) = c_1$, $v(0) = c_2$. This solution will satisfy (2) if (14) is satisfied and will be non-trivial provided c_1, c_2 are not both zero. For Case I at least two of the elements of $D(\lambda)$ must be different from zero. Without loss of generality assume that $U_1(u_2 v_2) \neq 0$, then by continuity it will not vanish for values of λ nearby. Put $c_1 \equiv -U_1(u_2 v_2)$, $c_2 \equiv U_1(u_1 v_1)$. Then $U_1(uv) \equiv 0$, and $U_2(uv) \equiv D(\lambda)$.

Define another solution (U, V) of (1) by

$$\begin{cases} U(x) \equiv -c_2 u_1(x) + c_1 u_2(x), \\ V(x) \equiv -c_2 v_1(x) + c_1 v_2(x). \end{cases} \tag{18}$$

Then (u, v), (U, V) will be linearly independent since

$$W \equiv \begin{vmatrix} u(0) & v(0) \\ U(0) & V(0) \end{vmatrix} = \begin{vmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{vmatrix} = c_1^2 + c_2^2 > 0.$$

Consider a solution $(\tilde{u}, \tilde{v})$ of (15) defined by

$$\begin{cases} \tilde{u}(x) \equiv u_0(x) + b_1 u(x) + b_2 U(x), \\ \tilde{v}(x) = v_0(x) + b_1 v(x) + b_2 V(x), \end{cases} \tag{19}$$

where (u_0, v_0) is a particular solution of (15) such that

$$u_0(0) = v_0(0) = 0.$$

Then

$$\begin{cases} U_1(\tilde{u}\tilde{v}) = U_1(u_0 v_0) - b_2 W, \\ U_2(\tilde{u}\tilde{v}) = U_2(u_0 v_0) + b_1 U_2(uv) + b_2 U_2(UV), \end{cases} \tag{20}$$

and $(\tilde{u}, \tilde{v})$ will satisfy (2) if

$$\begin{aligned} b_2 &= \frac{U_1(u_0 v_0)}{W}, \\ b_1 &= \frac{-U_1(u_0 v_0) U_2(UV) - U_2(u_0 v_0) W}{W U_2(uv)}, \end{aligned} \tag{21}$$

$$\begin{aligned} \tilde{u}(x) &= u_0(x) + \frac{U_1(u_0 v_0) U(x)}{W} - \frac{U_1(u_0 v_0) U_2(UV) + U_2(u_0 v_0) W}{W D(\lambda)} u(x), \\ \tilde{v}(x) &= v_0(x) + \frac{U_1(u_0 v_0) V(x)}{W} - \frac{U_1(u_0 v_0) U_2(UV) + U_2(u_0 v_0) W}{W D(\lambda)} v(x), \end{aligned} \tag{22}$$

since $U_2(uv) = D(\lambda)$.

Hence if the matrix of (11) is of rank two, (22) determines a solution

$(\tilde{u}, \tilde{v})$ analytic in λ for that range for which $U_1(u_2v_2) \neq 0$. For other ranges of values of λ within which $U_1(u_2v_2)$ vanishes we choose in its place another element of the determinant $D(\lambda)$ which differs from zero throughout that range. This process determines b_1, b_2 in a different fashion. Clearly since the roots of $U_1(u_2v_2)$ and of every other element of $D(\lambda)$ are isolated, each element being entire in λ, the solution $(\tilde{u}, \tilde{v})$ determined in either way for points of the λ-axis at which $U_1(u_2v_2) \neq 0$ and some other element, e.g., $U_2(u_1v_1) \neq 0$, will be identical since there is one and only one such solution by the existence theorem. $\therefore$ The proof is complete for Case I.

Case II: Let us first show that no root of $D(\lambda)$ is double for this case. Assuming that $U_1(u_2v_2) \neq 0$ for $\lambda = \lambda_n$ and defining c_1, c_2 as before, we have by differentiating as to λ, since $u(x)$, $v(x)$ of (17) are entire in λ and satisfy (1):

$$\begin{cases} N_1(u_\lambda v_\lambda \lambda) = v(x), \\ N_2(u_\lambda v_\lambda \lambda) = -u(x). \end{cases} \tag{23}$$

If $D(\lambda)$ has a double root $\lambda = \lambda_n$, then for this value of λ

$$U_2(uv) = 0, \quad U_2(u_\lambda v_\lambda) = 0; \qquad N_1(uv\lambda) = N_2(uv\lambda) = 0;$$
$$U_1(uv) = 0, \quad U_1(u_\lambda v_\lambda) = 0.$$

Whence by Lemmas II, III, Section I

$$\int_0^1 ([u(s)]^2 + [v(s)]^2) ds = 0.$$

This is impossible since $u(0) \equiv -U_1(u_2v_2) \not\equiv 0$. Therefore $D(\lambda)$ has no double root.

For values sufficiently near λ_n, $D(\lambda) \neq 0$ and the solution $(\tilde{u}, \tilde{v})$ of (15), (2) analytic in λ is given by (22). And by Lemma II, Section I

$$[u_0(x)v(x) - u(x)v_0(x)]_0^x = \int_0^x [v(s)\varphi(s) - u(s)\psi(s)] ds, \tag{24}$$
$$[u_0(x)V(x) - U(x)v_0(x)]_0^x = \int_0^x [V(s)\varphi(s) - U(s)\psi(s)] ds. \tag{25}$$

The left members vanish at the lower limit since $u_0(0)$ and $v_0(0)$ are equal to zero, hence solving we obtain

$$u_0(x) = -\frac{1}{W} \int_0^x \begin{vmatrix} u(x) & U(x) & 0 \\ u(s) & U(s) & \varphi(s) \\ v(s) & V(s) & \psi(s) \end{vmatrix} ds, \tag{26}$$

$$v_0(x) = -\frac{1}{W} \int_0^x \begin{vmatrix} v(x) & V(x) & 0 \\ u(s) & U(s) & \varphi(s) \\ v(s) & V(s) & \psi(s) \end{vmatrix} ds. \tag{27}$$

If $\lambda \to \lambda_n$, $\tilde{u}(x)$ approaches the value

$$u_0(x) + \frac{U_1(u_0v_0)}{W} U(x) - \frac{u(x)}{W} \left\{ \frac{\partial}{\partial \lambda} \frac{[U_1(u_0v_0)U_2(UV) + U_2(u_0v_0)W]}{D'(\lambda)} \right\}_{\lambda=\lambda_n} \tag{28}$$

provided only

$$U_1(u_0v_0)U_2(UV) + U_2(u_0v_0)W]_{\lambda=\lambda_n} = 0 \tag{29}$$

since $D'(\lambda) \neq 0$.

Again if (29) holds, $\tilde{v}(x)$ will approach a limit as $\lambda \to \lambda_n$. In such a case $(\tilde{u}, \tilde{v})$ will be continuous if put equal to the limit approached at $\lambda = \lambda_n$ and therefore analytic for all λ in the interval.

Since $W = -U_1(UV)$, from (26), (27) it follows that (29) is equivalent to

$$\int_0^1 \begin{vmatrix} \gamma_1u(1)+\delta_1v(1), & U_1(UV) \\ \gamma_2u(1)+\delta_2v(1), & U_2(UV) \\ u(s) & \end{vmatrix}, \begin{vmatrix} \gamma_1U(1)+\delta_1V(1), & U_1(UV) \\ \gamma_2U(1)+\delta_2V(1), & U_2(UV) \\ U(s) & \end{vmatrix}, \begin{vmatrix} 0 \\ \varphi(s) \\ \psi(s) \end{vmatrix}_{\lambda=\lambda_n} ds = 0. \tag{30}$$

The first element of the large determinant is the same as

$$\begin{vmatrix} \gamma_1u(1) + \delta_1v(1), & \gamma_1U(1) + \delta_1V(1) \\ \gamma_2u(1) + \delta_2v(1), & \gamma_2U(1) + \delta_2V(1) \end{vmatrix} - \begin{vmatrix} \alpha_1u(0) + \beta_1v(0), & \alpha_1U(0) + \beta_1V(0) \\ \alpha_2u(0) + \beta_2v(0), & \alpha_2U(0) + \beta_2V(0) \end{vmatrix}$$

since for $\lambda = \lambda_n$, $U_1(uv) = U_2(uv) = 0$, or simplifying further, this element reduces to $(\gamma\delta)W - (\alpha\beta)W = 0$.

Thus a sufficient condition for the existence of a solution $(\tilde{u}, \tilde{v})$ of (15), (2) analytic for all λ when $D(\lambda)$ is of rank one is that (16) be satisfied for all n such that $D(\lambda_n) = 0$, and the theorem is proved for Case II.

Case III. When every element of $D(\lambda)$ vanishes for some $\lambda = \lambda_k$, it is obvious that $D'(\lambda_k)$ also vanishes. Let us show that $D''(\lambda_k) \neq 0$ for such a case.

By the same reasoning as was used to prove $D'(\lambda) \neq 0$ for $\lambda = \lambda_n$ in Case II we show that if $U_1(u_{1\lambda}v_{1\lambda})$ and $U_2(u_{1\lambda}v_{1\lambda})$ both vanish at $\lambda = \lambda_k$, then

$$\int_0^1 ([u_1(x)]^2 + [v_1(x)]^2)dx = 0,$$

which is impossible since $u_1(0) = 1$. Hence $U_1(u_{1\lambda}v_{1\lambda})$ and $U_2(u_{1\lambda}v_{1\lambda})$ cannot both vanish at $\lambda = \lambda_k$. Similarly $U_1(u_{2\lambda}v_{2\lambda})$ and $U_2(u_{2\lambda}v_{2\lambda})$ cannot both vanish there. Without loss of generality we may assume

$$U_1(u_{2\lambda}v_{2\lambda}) \neq 0, \qquad \lambda = \lambda_k. \tag{31}$$

Now define a solution of (1) by

$$\begin{cases} u(x) \equiv u_1(x) + \dfrac{c_1}{c_2}u_2(x), \\ v(x) \equiv v_1(x) + \dfrac{c_1}{c_2}v_2(x), \end{cases} \tag{32}$$

and choose

$$\frac{c_1}{c_2} \equiv -\frac{U_1(u_1v_1)}{U_1(u_2v_2)}.$$

For values of λ sufficiently near λ_k, $U_1(u_2v_2) \neq 0$, otherwise (31) would be violated by Rolle's Theorem. Hence for all λ in the interval considered

$$U_1(uv) \equiv U_1(u_1v_1) - \frac{U_1(u_1v_1)}{U_1(u_2v_2)} U_1(u_2v_2) \equiv 0.$$

Consequently for all such λ, $U_1(u_\lambda v_\lambda) \equiv 0$.

Again

$$U_2(uv) = U_2(u_1v_1) - \frac{U_1(u_1v_1)}{U_1(u_2v_2)} U_2(u_2v_2) = \frac{-D(\lambda)}{U_1(u_2v_2)}, \qquad \lambda \neq \lambda_k.$$

At $\lambda = \lambda_k$,

$$U_2(uv) = -\frac{D'(\lambda)}{U_1(u_{2\lambda}v_{2\lambda})} = 0.$$

And

$$U_2(u_\lambda v_\lambda) = -\frac{D'(\lambda)}{U_1(u_2v_2)} + \frac{D(\lambda)U_1(u_{2\lambda}v_{2\lambda})}{[U_1(u_2v_2)]^2}, \qquad \lambda \neq \lambda_k.$$

For $\lambda = \lambda_k$,

$$U_2(u_\lambda v_\lambda) = -\frac{D''(\lambda)}{U_1(u_{2\lambda}v_{2\lambda})} + \frac{D(\lambda)U_1(u_{2\lambda\lambda}v_{2\lambda\lambda}) + D'(\lambda)U_1(u_{2\lambda})}{2U_1(u_2v_2)U_1(u_{2\lambda}v_{2\lambda})}.$$

This vanishes if $D''(\lambda) = 0$ at $\lambda = \lambda_k$. Applying the argument a third time we obtain

$$\int_0^1 (u^2 + v^2)dx = 0,$$

which is absurd since by (32) $u(0) = 1$.

$$\therefore \quad D''(\lambda) \neq 0 \text{ at } \lambda = \lambda_k.$$

Since for Case III when $\lambda = \lambda_k$ every solution of (1) satisfies (2), if we define a solution of (15) by

$$\begin{cases} \tilde{u}(x) \equiv u_0(x) + b_1u_1(x) + b_2u_2(x), \\ \tilde{v}(x) \equiv v_0(x) + b_1v_1(x) + b_2v_2(x), \end{cases} \tag{33}$$

where $u_0(0) = v_0(0) = 0$ defines a particular solution of (15), then $U_1(\tilde{u}\tilde{v}) = U_1(u_0v_0)$ and $U_2(\tilde{u}\tilde{v}) = U_2(u_0v_0)$. If these vanish b_1, b_2 will be arbitrary. Again by using results similar to (24), (25), (26), (27) we have, since

$$\begin{vmatrix} u_1(0) & v_1(0) \\ u_2(0) & v_2(0) \end{vmatrix} = 1,$$

$$U_1(u_0v_0) \equiv -\int_0^1 \begin{vmatrix} \gamma_1u_1(1) + \delta_1v_1(1), & \gamma_1u_2(1) + \delta_1v_2(1), & 0 \\ u_1(s), & u_2(s), & \varphi(s) \\ v_1(s), & v_2(s), & \psi(s) \end{vmatrix} ds,$$

$$U_2(u_0v_0) \equiv -\int_0^1 \begin{vmatrix} \gamma_2u_1(1) + \delta_2v_1(1), & \gamma_2u_2(1) + \delta_2v_2(1), & 0 \\ u_1(s), & u_2(s), & \varphi(s) \\ v_1(s), & v_2(s), & \psi(s) \end{vmatrix} ds.$$

For $\lambda = \lambda_k$, (u_1, v_1), (u_2, v_2) are solutions of (1), (2). Hence if we assume

that (16) holds for all solutions (u_n, v_n) of (1), (2), then $(\tilde{u}, \tilde{v})$ will satisfy (2) for b_1, b_2 arbitrary.

We wish to choose values for them such as to make the solution $(\tilde{u}, \tilde{v})$, now analytic $\lambda \neq \lambda_k$, continuous at $\lambda = \lambda_k$.

Since every solution of (1) may be expressed as a linear combination of u_1, u_2, v_1, v_2, we may put (33) in the form

$$\begin{cases} \tilde{u}(x) \equiv u_0(x) + d_1 u(x) + d_2 u_2(x), \\ \tilde{v}(x) \equiv v_0(x) + d_1 v(x) + d_2 v_2(x), \end{cases} \tag{34}$$

in which (u, v) is defined by (32) and c_1/c_2 has the same value as before. (u, v) and (u_2, v_2) are linearly independent since

$$\begin{vmatrix} u(0) & v(0) \\ u_2(0) & v_2(0) \end{vmatrix} = \begin{vmatrix} 1 & c_1/c_2 \\ 0 & 1 \end{vmatrix} \neq 0.$$

As $U_1(uv) \equiv 0$, we have

$$U_1(\tilde{u}\tilde{v}) = U_1(u_0 v_0) + d_2 U_1(u_2 v_2),$$

which will vanish for all λ in the interval considered if

$$d_2 \equiv -\frac{U_1(u_0 v_0)}{U_1(u_2 v_2)}.$$

Again

$$U_2(\tilde{u}\tilde{v}) = U_2(u_0 v_0) + d_1 U_2(uv) + d_2 U_2(u_2 v_2) = 0,$$

provided

$$d_1 = -\frac{U_2(u_0 v_0) U_1(u_2 v_2) + U_1(u_0 v_0) U_2(u_2 v_2)}{U_1(u_2 v_2) U_2(uv)}.$$

For $\lambda \neq \lambda_k$,

$$U_2(uv) = -\frac{D(\lambda)}{U_1(u_2 v_2)}$$

and

$$d_1 = \frac{U_2(u_0 v_0) U_1(u_2 v_2) + U_1(u_0 v_0) U_2(u_2 v_2)}{D(\lambda)}.$$

Hence for $\lambda = \lambda_k$ in a sufficiently small interval, $D(\lambda) \neq 0$ and $U_1(u_2, v_2) \neq 0$,

$$\tilde{u}(x) = u_0(x) + \frac{U_2(u_0 v_0) U_1(u_2 v_2) + U_1(u_0 v_0) U_2(u_2 v_2)}{D(\lambda)} u(x) - \frac{U_1(u_0 v_0)}{U_1(u_2 v_2)} u_2(x). \tag{35}$$

If we let λ approach λ_k we get as the value approached by the right member:

$$u_0(x) - \frac{U_1(u_{0\lambda} v_{0\lambda})}{U_1(u_{2\lambda} v_{2\lambda})} u_2(x) + u(x) \frac{\dfrac{\partial^2}{\partial \lambda^2}\left[U_2(u_0 v_0) U_1(u_2 v_2) + U_1(u_0 v_0) U_2(u_2 v_2)\right]}{D''(\lambda)}, \qquad \lambda = \lambda_k.$$

Evidently $u(x)$ approaches a limit since

$$\frac{c_1}{c_2} \to -\frac{U_1(u_{1\lambda} v_{1\lambda})}{U_1(u_{2\lambda} v_{2\lambda})}.$$

Also

$$\frac{\partial}{\partial\lambda}[U_2(u_0v_0)U_1(u_2v_2) + U_1(u_0v_0)U_2(u_2v_2)] = 0, \qquad \lambda = \lambda_k,$$

since each factor of both terms vanishes there. Also $D'(\lambda_k) = 0$.

Hence $\tilde{u}(x)$ approaches a limit as $\lambda \to \lambda_k$, and similarly $\tilde{v}(x)$ does also. Thus we have a solution $(\tilde{u}, \tilde{v})$ analytic for all λ in a small enough interval about $\lambda = \lambda_k$, which satisfies (2). By extending the reasoning as in Case I the proof is completed.

THEOREM II: *If for* $n = 0, \pm 1, \pm 2, \cdots$

$$\int_0^1 \begin{vmatrix} u_n(s) & \varphi(s) \\ v_n(s) & \psi(s) \end{vmatrix} ds = 0,$$

then $\varphi(x) \equiv \psi(x) \equiv 0$.

The proof will be merely outlined here since it is given in full in a recent article by Professor W. A. Hurwitz.

Let the solution of (15), (2) shown by the previous theorem to exist be represented by

$$\begin{aligned} u(x) &\equiv y_0(x) + \lambda y_1(x) + \lambda^2 y_2(x) + \cdots, \\ v(x) &\equiv z_0(x) + \lambda z_1(x) + \lambda^2 z_2(x) + \cdots. \end{aligned} \tag{36}$$

Since λ is restricted to real values we have but real functions of x with which to deal. Putting (36) in (15), (2) and equating coefficients of λ^k we get sets of differential equations and boundary conditions satisfied by (y_m, z_m), $m = 0, 1, 2, \cdots$. By eliminating $a(x)$, $b(x)$ from two different sets and using Lemma III, Section I, one gets

$$\int_0^1 (y_m y_n + z_m z_n)dx = (\int_0^1 y_{m-1}y_{n+1} + z_{m-1}z_{n+1})dx = W_{m+n},$$

where

$$W_r \equiv \int_0^1 (y_r y_0 + z_r z_0)dx.$$

Then from the inequality

$$\begin{vmatrix} y_{m+1}(x) & y_{m+1}(\xi) \\ y_{m-1}(x) & y_{m-1}(\xi) \end{vmatrix}^2 + \begin{vmatrix} y_{m+1}(x) & z_{m+1}(\xi) \\ y_{m-1}(x) & z_{m-1}(\xi) \end{vmatrix}^2 + \begin{vmatrix} z_{m+1}(x) & y_{m+1}(\xi) \\ z_{m-1}(x) & y_{m-1}(\xi) \end{vmatrix}^2 + \begin{vmatrix} z_{m+1}(x) & z_{m+1}(\xi) \\ z_{m-1}(x) & z_{m-1}(\xi) \end{vmatrix}^2 \geqq 0, \tag{37}$$

for x and ξ in the interval from zero to one, by integrating as to x and ξ successively from 0 to 1 and simplifying, we have

$$W_{2m-2}W_{2m+2} - W_{2m}^2 \geqq 0. \tag{38}$$

Next comes the lemma that some $W_{2k} = 0$. Assume no $W_{2k} = 0$. Multiplying the series of (36) by $y_0(x)$, $z_0(x)$ respectively, adding and integrating

we obtain the uniformly convergent series

$$W_0 + \lambda W_1 + \lambda^2 W_2 + \cdots. \tag{39}$$

This converges absolutely as also does

$$|W_0| + \lambda^2 |W_2| + \lambda^4 |W_4| + \cdots. \tag{40}$$

Using (38), i.e.

$$\frac{W_0}{W_2} \geqq \frac{W_2}{W_4} \geqq \frac{W_4}{W_6} \geqq \cdots,$$

for $\lambda = \sqrt{\left|\frac{W_0}{W_2}\right|}$ in this we get an absurdity:

$|W_0| + |W_0| + |W_0| + \cdots$ convergent. Thus the lemma is proved, i.e.

$$\int_0^1 (y_k^2 + z_k^2) dx = 0, \qquad \text{or} \qquad y_k(x) \equiv z_k(x) \equiv 0. \tag{41}$$

Thence from the differential equations

$$y_{k-1}(x) \equiv y_{k-2}(x) \equiv \cdots \equiv y_0(x) \equiv 0,$$

and similarly for z and also $\psi(x) \equiv \varphi(x) \equiv 0$. Q.e.d.

Section III. Asymptotic Formulæ.

THEOREM III: For $|\lambda|$ large a solution of (1) defined by $u(0) = \alpha$, $v(0) = \beta$ takes the form

$$\begin{cases} u(x) = \alpha \cos \xi + \beta \sin \xi + O\left(\frac{1}{\lambda}\right), \\ v(x) = \beta \cos \xi - \alpha \sin \xi + O\left(\frac{1}{\lambda}\right), \end{cases} \tag{42}$$

and its partial derivatives, the form

$$\begin{cases} u_\lambda(x) = \beta x \cos \xi - \alpha x \sin \xi + O\left(\frac{1}{\lambda}\right), \\ v_\lambda(x) = -\alpha x \cos \xi - \beta x \sin \xi + O\left(\frac{1}{\lambda}\right), \end{cases} \tag{43}$$

where

$$\xi \equiv \lambda x + \tfrac{1}{2} \int_0^x [a(s) + b(s)] ds. \tag{44}$$

I give an outline of a proof analogous to that used in Professor Hurwitz's recent article.

Assume

$$\begin{cases} u(x) \equiv U + \left(1 + \frac{a(x)}{\lambda}\right)(\alpha \cos \xi + \beta \sin \xi), \\ v(x) \equiv V + \left(1 + \frac{b(x)}{\lambda}\right)(\beta \cos \xi - \alpha \sin \xi). \end{cases} \tag{45}$$

Putting in (1) we get

$$\begin{cases} N_1(UV\lambda) = \dfrac{\varphi}{\lambda}, \\ N_2(UV\lambda) = \dfrac{\psi}{\lambda}, \end{cases} \tag{46}$$

where φ, ψ, φ_λ, ψ_λ are $O(1)$ as regards λ.

Using as multipliers $\cos \lambda x$, $-\sin \lambda x$, adding, integrating from 0 to x, and combining this with the result of repeating the procedure with $\sin \lambda x$, $\cos \lambda x$ as multipliers we get

$$\begin{cases} U(x) = \dfrac{F}{\lambda} + \int_0^x [K_{11}U(s) + K_{12}V(s)]ds, \\ V(x) = \dfrac{G}{\lambda} + \int_0^x [K_{21}U(s) + K_{22}V(s)]ds, \end{cases} \tag{47}$$

where F, G and the K's are $O(1)$, and F_λ, G_λ, K_λ are also.

$$\begin{aligned} \therefore \quad U_\lambda(x) &= \frac{H}{\lambda} + \int_0^x [K_{31}U(s) + K_{32}V(s) + K_{33}U_\lambda(s) + K_{34}V_\lambda(s)]ds, \\ V_\lambda(x) &= \frac{J}{\lambda} + \int_0^x [K_{41}U(s) + K_{42}V(s) + K_{43}U_\lambda(s) + K_{44}V_\lambda(s)]ds, \end{aligned} \tag{48}$$

in which H, J, and the K's of (48) are $O(1)$. By the process of successive approximations we obtain from (47), (48),

$$U(x) = O\left(\frac{1}{\lambda}\right), \quad V(x) = O\left(\frac{1}{\lambda}\right), \quad U_\lambda(x) = O\left(\frac{1}{\lambda}\right), \quad V_\lambda(x) = O\left(\frac{1}{\lambda}\right). \tag{49}$$

From (45), (49) we see that (42) is true. Also by differentiating (45) as to λ and using (49) the rest of the proof is obvious.

COROLLARY: For $|\lambda|$ large $D(\lambda)$, $D'(\lambda)$ take the forms

$$\begin{cases} D(\lambda) = \sqrt{A^2 + B^2} \sin [\xi(1) + \varphi] + C + O\left(\dfrac{1}{\lambda}\right), \\ D'(\lambda) = \sqrt{A^2 + B^2} \cos [\xi(1) + \varphi] + O\left(\dfrac{1}{\lambda}\right), \end{cases} \tag{50}$$

where

$$A \equiv (\alpha\gamma) + (\beta\delta), \qquad B \equiv (\alpha\delta) + (\gamma\beta), \qquad C = 2(\alpha\beta), \tag{51}$$

and φ is defined by

$$\cos \varphi \equiv \frac{A}{\sqrt{A^2 + B^2}}, \qquad \sin \varphi \equiv \frac{B}{\sqrt{A^2 + B^2}}. \tag{52}$$

By Lemma VI, Section I, A and B cannot both vanish. From Theorem III

$$\begin{cases} u_1(x) = \cos\xi + O\left(\frac{1}{\lambda}\right), \\ v_1(x) = -\sin\xi + O\left(\frac{1}{\lambda}\right), \\ u_2(x) = \sin\xi + O\left(\frac{1}{\lambda}\right), \\ v_2(x) = \cos\xi + O\left(\frac{1}{\lambda}\right). \end{cases} \tag{53}$$

Putting these in (11) we get

$$D(\lambda) = A\sin\xi(1) + B\cos\xi(1) + (\alpha\beta) + (\gamma\delta) + O\left(\frac{1}{\lambda}\right). \tag{54}$$

Again using the asymptotic values for $u_{1\lambda}(x)$, $v_{1\lambda}(x)$, etc., from (53) in the identity

$$D'(\lambda) \equiv \begin{vmatrix} U_1(u_{1\lambda}v_{1\lambda}) & U_1(u_{2\lambda}v_{2\lambda}) \\ U_2(u_1v_1) & U_2(u_2v_2) \end{vmatrix} + \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_{1\lambda}v_{1\lambda}) & U_2(u_{2\lambda}v_{2\lambda}) \end{vmatrix}$$

we derive

$$D'(\lambda) = A\cos\xi(1) - B\sin\xi(1) + O\left(\frac{1}{\lambda}\right),$$

and (50) follows by (3), (51), (52).

To show $D(\lambda)$ has roots, no matter how large $|\lambda|$ is, consider the

THEOREM IV: *If l_n is a principal parameter value for the system* (1) *and the boundary conditions*

$$\begin{cases} \alpha_1u(0) + \beta_1v(0) = 0, \\ \gamma_1u(1) + \delta_1v(1) = 0, \end{cases} \tag{55}$$

then there exist exactly two roots of $D(\lambda)$ in the intervals:

Case I. (l_{2p}, l_{2p+2}), $p = 0, \pm 1, \pm 2, \cdots$, if $P_{21} > 0$ for $\lambda = l_1$;

Case II. (l_{2p-1}, l_{2p+1}), $p = 0, \pm 1, \pm 2, \cdots$, if $P_{21} < 0$ for $\lambda = l_1$.*

Define

$$\begin{cases} B_1(x) \equiv \alpha_1u(x) + \beta_1v(x), \\ B_2(x) \equiv \alpha_2u(x) + \beta_2v(x), \\ P_1(x) \equiv -\gamma_1u(x) - \delta_1v(x), \\ P_2(x) \equiv -\gamma_2u(x) - \delta_2v(x), \end{cases} \tag{56}$$

and determine two solutions, $[u_1(x\lambda), v_1(x\lambda)]$, $[u_2(x\lambda), v_2(x\lambda)]$, of (1) such that

$$\begin{cases} B_{11}(0) = 0, & B_{21}(0) = 1, \\ B_{12}(0) = 1, & B_{22}(0) = 0; \end{cases} \tag{57}$$

in which the second subscript refers to the solution involved. Without

* I have followed the method of proof used by Professor Birkhoff in an article published in the *Transactions* in 1909 entitled, "Existence and Oscillation Theorem for a Certain Boundary Value Problem."

loss of generality we may assume that the coefficients α_1, β_1, γ_1, δ_1 have been divided by a constant such that

$$(\alpha\beta) = (\gamma\delta) = 1. \tag{58}$$

Then we prove the

LEMMA I: *At a simple value λ_n of λ, $D(\lambda)$ changes sign in such a way that $D'(\lambda)$ has the sign of $-P_{11}$ or P_{22}, where*

$$\begin{cases} P_{11} \equiv -\gamma_1 u_1(1) - \delta_1 v_1(1), \\ P_{22} \equiv -\gamma_2 u_2(1) - \delta_2 v_2(1). \end{cases} \tag{59}$$

The solutions (u_1,v_1), (u_2,v_2) defined by (57) are linearly independent since

$$\begin{vmatrix} u_1(0) & v_1(0) \\ u_2(0) & v_2(0) \end{vmatrix} = \begin{vmatrix} -\beta_1 & \alpha_1 \\ \beta_2 & -\alpha_2 \end{vmatrix} = (\alpha\beta) = -1 \tag{60}$$

and hence by a relation similar to Abel's

$$|u(x), \quad v(x)| = \text{constant} \neq 0.$$

Thus

$$B_{11}(x)B_{22}(x) - B_{12}(x)B_{21}(x) = -1, \qquad 0 \leqq x \leqq 1. \tag{61}$$

Likewise

$$P_{11}(x)P_{22}(x) - P_{12}(x)P_{21}(x) = -1, \qquad 0 \leqq x \leqq 1. \tag{62}$$

Also any solution of (1) may be written

$$u(x) = c_1u_1(x) + c_2u_2(x),$$
$$v(x) = c_1v_1(x) + c_2v_2(x).$$

If this is to satisfy (2) we must have

$$D(\lambda) = \begin{vmatrix} -P_{11}, & 1 - P_{12} \\ 1 - P_{21}, & -P_{22} \end{vmatrix} = 0. \tag{63}$$

We have shown that the necessary condition for a double value of λ is that $D(\lambda)$ be of rank zero, i.e.,

$$\begin{cases} P_{11} = P_{22} = 0, \\ P_{21} = P_{12} = 1. \end{cases} \tag{64}$$

This is obviously also sufficient.

Since

$$N_1(u_1v_1\lambda) = N_2(u_1v_1\lambda) = 0 \tag{65}$$

and

$$N_1(u_{1\lambda}v_{1\lambda}\lambda) = v_1, \qquad N_2(u_{1\lambda}v_{1\lambda}\lambda) = -u, \tag{66}$$

by Lemma II, Section I,

$$u_{1\lambda}v_1 - u_1v_{1\lambda} = \int_0^x (u_1^2 + v_1^2)ds. \tag{67}$$

Similarly

$$u_{2\lambda}v_2 - u_2v_{2\lambda} = \int_0^x (u_2^2 + v_2^2)ds. \tag{68}$$

Again from

$$N_1(u_2v_2\lambda) = N_2(u_2v_2\lambda) = 0 \tag{69}$$

and (66) by the same Lemma

$$u_{1\lambda}v_2 - u_2v_{1\lambda} = \int_0^x (u_1u_2 + v_1v_2)ds. \tag{70}$$

Similarly

$$u_{2\lambda}v_1 - u_1v_{2\lambda} = \int_0^x (u_1u_2 + v_1v_2)ds. \tag{71}$$

From (67), (70)

$$u_{1\lambda}(x) = \int_0^x [(u_1^2 + v_1^2)u_2(x) - (u_1u_2 + v_1v_2)u_1(x)]ds. \tag{72}$$

Similarly

$$v_{1\lambda}(x) = \int_0^x [(u_1^2 + v_1^2)v_2(x) - (u_1u_2 + v_1v_2)v_1(x)]ds. \tag{73}$$

Also from (68), (71)

$$u_{2\lambda}(x) = \int_0^x [(u_1u_2 + v_1v_2)u_2(x) - (u_2^2 + v_2^2)u_1(x)]ds, \tag{74}$$

$$v_{2\lambda}(x) = \int_0^x [(u_1u_2 + v_1v_2)\, v_2(x) - (u_2^2 + v_2^2)\, v_1(x)]ds. \tag{75}$$

Now from (62), (63)

$$D(\lambda) \equiv P_{12} + P_{21} - 2. \tag{76}$$

Hence by (72), (73), (74), (75), (76)

$$D'(\lambda) = P_{12\lambda} + P_{21\lambda}$$

or

$$\begin{aligned} D'(\lambda) = \int_0^1 [&P_{22}u_1^2 + (P_{12} - P_{21})u_1u_2 - P_{11}u_2^2 \\ &+ P_{22}v_1^2 + (P_{12} - P_{21})v_1v_2 - P_{11}v_2^2]ds. \end{aligned} \tag{77}$$

The integrand consists of two quadratic forms whose discriminant is

$$(P_{12} - P_{21})^2 + 4P_{11}P_{22} \tag{78}$$

or by (62)

$$(P_{12} + P_{21})^2 - 4. \tag{79}$$

This vanishes when $D(\lambda)$ does. For a simple root of $D(\lambda) P_{11}$, P_{22} cannot both vanish since then by (78) and (63), (64) would be satisfied. Again neither form can vanish because (u_1,v_1) and (u_2,v_2) are linearly independent. Thus the Lemma is proved.

LEMMA II: *At a double value of* λ, *say* λ_k, $D(\lambda)$ *maintains a negative sign.*

From (64)

$$\Delta P_{11} = \overline{P}_{11}, \quad \Delta P_{22} = \overline{P}_{22}, \quad 1 + \Delta P_{12} = \overline{P}_{12}, \quad 1 + \Delta P_{21} = \overline{P}_{21}, \tag{80}$$

in which

$$\overline{P} \equiv P(\lambda + \Delta\lambda), \qquad \lambda = \lambda_k. \tag{81}$$

Then from (62)

$$\overline{P}_{11}\overline{P}_{22} - \overline{P}_{12}\overline{P}_{21} = -1$$

or

$$\Delta P_{11}\Delta P_{22} - (1 + \Delta P_{12})(1 + \Delta P_{21}) = -1, \tag{82}$$

and by (76)

$$\Delta D = \Delta P_{12} + \Delta P_{21} = \Delta P_{11}\Delta P_{22} - \Delta P_{12}\Delta P_{21}. \tag{83}$$

Clearly

$$\Delta P_{11} \equiv -\gamma_1 \Delta u_1(1) - \delta_1 \Delta v_1(1),$$

and so for the others. Also

$$N_1(\Delta u, \Delta v, \lambda) = \Delta\lambda \bar{v},$$
$$N_2(\Delta u, \Delta v, \lambda) = -\Delta\lambda \bar{u}.$$

Using these for (u_1,v_1), (u_2,v_2) we get results for $\frac{\Delta u_1}{\Delta\lambda}, \frac{\Delta u_2}{\Delta\lambda}, \frac{\Delta v_1}{\Delta\lambda}, \frac{\Delta v_2}{\Delta\lambda}$ which differ little from the right members of equations (72) to (75). Putting these values in (83) and omitting in each term infinitesimals of order greater than two, we obtain

$$\Delta D = \left[\int_0^1 (u_1u_2 + v_1v_2)\Delta\lambda ds\right]^2 - \int_0^1 (u_1^2 + v_1^2)\Delta\lambda ds \int_0^1 (u_2^2 + v_2^2)\Delta\lambda ds.$$

By Lemma IV, Section I, $\Delta D < 0$, since the solutions are real, and $D(\lambda)$ preserves a negative sign at $\lambda = \lambda_k$.

LEMMA III: *$D(\lambda)$ has the same sign as P_{21} at the values $\lambda = l_n$, $n = 0, \pm 1, \pm 2, \cdots$, unless $P_{21} = 1$, when $D(\lambda) = 0$.*

Since $u_1(0) = -\beta$, $v_1(0) = \alpha$ (Cf. (60)), (u_1,v_1) is the only solution except for a constant factor which satisfies (55), (1) for $\lambda = l_0, l_{\pm 1}, l_{\pm 2}, \cdots$ but for no others. (See Professor Hurwitz's article.) These values separate the λ-axis into the intervals

$$\cdots (l_{-n}, l_{-n+1}), \quad \cdots (l_{-1}, l_0), \quad (l_0, l_1), \quad \cdots (l_n, l_{n+1}), \quad \cdots \tag{84}$$

By (62) at $\lambda =$ one of these values, say l_r,

$$P_{12}P_{21} = 1. \tag{85}$$

And by (76)

$$D(l_r) = \frac{1}{P_{21}}(1 - P_{21})^2, \qquad r = 0, \pm 1, \pm 2, \cdots. \tag{86}$$

And the rest of the proof is evident.

Professor Hurwitz has shown that P_{11} has no double roots, also that for the system (1) and

$$\begin{cases} B_{11}(0) = 0, \\ P_{21}(1) = 0, \end{cases} \tag{87}$$

P_{21} has no double roots. Hence P_{11}, P_{21} change sign when they vanish. Again since

$$P_{11}P_{21\lambda} - P_{11\lambda}P_{21} \equiv u_1v_{1\lambda} - v_1u_{1\lambda}\Big]_{x=1},$$

by (67) this cannot vanish but is negative. From these facts it is easy to show that the roots of P_{11}, P_{21} separate each other as well as do those of

$P_{11\lambda}$, $P_{21\lambda}$. Clearly then P_{21} alternates in sign at the values $\cdots l_{-2}, l_{-1}, l_0, l_1, l_2, \cdots$ and we have two cases:

$$\text{Case I: } \left\{ \begin{array}{l} D(\lambda) \geqq 0 \text{ at } \lambda = l_{\pm 1}, l_{\pm 3}, \cdots \\ D(\lambda) < 0 \text{ at } \lambda = l_0, \quad l_{\pm 2}, \cdots \end{array} \right\} \text{ when } P_{21} > 0 \text{ for } \lambda = l_1;$$

$$\text{Case II: } \left\{ \begin{array}{l} D(\lambda) < 0 \text{ at } \lambda = l_{\pm 1}, l_{\pm 3}, \cdots \\ D(\lambda) \geqq 0 \text{ at } \lambda = l_0, \quad l_{\pm 2}, \cdots \end{array} \right\} \text{ when } P_{21} < 0 \text{ for } \lambda = l_1.$$

There must obviously exist roots of $D(\lambda)$ as follows:

In Case I—at least two values λ_n, λ_{n+1} in each double interval

$$(l_{2p}, l_{2p+2}), \qquad p = 0, \pm 1, \pm 2, \cdots,$$

such that

$$l_{2p} < \lambda_n \leqq l_{2p+1} \leqq \lambda_{n+1} < l_{2p+2} \tag{88}$$

and at least one such that

$$l_1 \leqq \lambda_r < l_2.$$

In Case II—at least two values λ_m, λ_{m+1} in each double interval

$$(l_{2p-1}, l_{2p+1}), \qquad p = 0, \pm 1, \pm 2, \cdots,$$

such that

$$l_{2p-1} < \lambda_m \leqq l_{2p} \leqq \lambda_{m+1} < l_{2p+1}. \tag{89}$$

To show that there are exactly two roots in (l_{2p}, l_{2p+2}) we make use of the separative property of the roots of P_{11}, P_{21} and their derivatives. Obviously there must be an even number of roots, since we count a double root as two. If a double root occurs it must fall at l_{2p+1} by (64) and Lemma III. Then by Lemmas II and I there can be no root elsewhere. If there is no double root, there must be less than four roots, for otherwise there would be at least two in one of the intervals (l_{2p}, l_{2p+1}), (l_{2p+1}, l_{2p+2}), which violates Lemma I.

Similarly we can show that there is exactly one root in the interval (l_1, l_2) and exactly two for Case II in the intervals (l_{2p-1}, l_{2p+1}) provided we count a double root at l_1 once each in the intervals (l_0, l_1), (l_1, l_2). Thus the Theorem is proved.

COROLLARY. *For* $|\lambda|$ *large,* $O(1/\lambda_n) = O(1/n\pi)$, $|n|$ *large.*

Professor Hurwitz has shown that for $|\lambda|$ large, $l_n = n\pi + \theta_1 + O(1/n)$, where θ_1 is a constant. Thus the intervals (l_n, l_{n+2}) for $|\lambda|$, $|n|$ large are of length 2π to within $O(1/n)$. In each of these intervals for n odd or even according to Case I or II by the Theorem there exist just two values λ_m, λ_{m+1} of λ which are roots of $D(\lambda)$. Since in any finite interval of the λ-axis there are but a finite number of roots l_r or λ_m, for $|n|$ large enough we have

$$\lambda = l_{n+s} - k, \qquad 0 \leqq |k| \leqq \pi,$$

where s is a finite integer positive or negative, provided we begin to count the l's and λ's so that $l_0 \geqq 0$, $\lambda_0 \geqq 0$ and $l_{-1} < 0$, $\lambda_{-1} < 0$. Clearly then

$$\lambda_n = n\pi + s\pi - k + \theta_1 + O\left(\frac{1}{n}\right) \tag{90}$$

and since $n\pi$ is the dominant part,

$$O\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{n\pi}\right), \text{ q.e.d.}$$

THEOREM V: *If (u_n, v_n) corresponds to λ_n defined as in the previous Corollary, then this solution of* (1), (2) *takes the asymptotic form*

$$\begin{cases} u_n(x) = \sin\left[n\pi x + P(x)\right] + O\left(\frac{1}{n}\right), \\ v_n(x) = \cos\left[n\pi x + P(x)\right] + O\left(\frac{1}{n}\right), \end{cases} \tag{91}$$

when $|n|$ is large.

By the Corollary of Theorem III $D(\lambda)$ will have a root when

$$\sin\left[\xi(1) + \varphi\right] = -\frac{C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right) \tag{92}$$

or when

$$\xi(1) + \varphi = \sin^{-1}\frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{n}\right), \tag{93}$$

since by Lemma V, Section I, $|C| \leqq \sqrt{A^2 + B^2}$, and by Theorem IV the right member of (92) must be numerically $\leqq 1$, also

$$\sin^{-1}\left[\frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right)\right] = \sin^{-1}\frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{\lambda}\right)$$

and for $\lambda = \lambda_n$,

$$O\left(\frac{1}{\lambda}\right) = O\left(\frac{1}{n\pi}\right) = O\left(\frac{1}{n}\right).$$

Hence from (93), since by (58) $C > 0$, $D(\lambda) = 0$ when

$$\lambda + \int_0^1 \frac{a(x) + b(x)}{2}\,dx + \varphi = \sin^{-1}\frac{-C}{\sqrt{A^2 + B^2}} + O\left(\frac{1}{n}\right)$$

and for m odd

$$\begin{cases} \lambda_n = m\pi + \psi - \int_0^1 \frac{a(x) + b(x)}{2}\,dx - \varphi + O\left(\frac{1}{n}\right), \\ \lambda_{n+1} = (m+1)\pi - \psi - \int_0^1 \frac{a(x) + b(x)}{2}\,dx - \varphi + O\left(\frac{1}{n}\right), \end{cases} \tag{94}$$

in which ψ is the angle in radian measure between 0 and $\pi/2$ inclusive

whose sine is $|C|/\sqrt{A^2+B^2}$. Clearly (94) may be expressed as

$$\lambda_n = m\pi + (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2}\,dx - \varphi + O\left(\frac{1}{n}\right), \text{ all } n, \quad (95)$$

where $|n|$, $|\lambda_n|$ are large and $m \equiv n + s$. From (42)

$$u(x) = \sin(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

$$v(x) = \cos(\xi - \theta_0) + O\left(\frac{1}{\lambda}\right),$$

provided we choose

$$\alpha \equiv -\sin\theta_0, \qquad \beta \equiv \cos\theta_0.$$

This is always possible if we divide $u(0)$, $v(0)$ by $\sqrt{[u(0)]^2 + [v(0)]^2}$ and call the quotients α, β respectively. Then by (95)

$$u_n(x) = \sin\left[m\pi x + \theta_2 x + \tfrac{1}{2}\int_0^x [a(s)+b(s)]ds - \theta_0\right] + O\left(\frac{1}{n}\right),$$

$$v_n(x) = \cos\left[m\pi x + \theta_2 x + \tfrac{1}{2}\int_0^x [a(s)+b(s)]ds - \theta_0\right] + O\left(\frac{1}{n}\right),$$

where

$$\theta_2 \equiv (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2}\,dx - \varphi$$

and the formulæ (91) follow, provided we define

$$P(x) \equiv \theta_2 x + s\pi x + \int_0^x \frac{a(s)+b(s)}{2}\,ds - \theta_0,$$

since $m \equiv n + s$, $\theta_2 \equiv -k + \theta_1$ as shown by (90).

COROLLARY: *If we normalize* (u_n, v_n), *it will have the same form* (91).

Section IV. Expansion Theorems.

THEOREM VI: *If two otherwise arbitrary functions* $f(x)$, $g(x)$ *satisfy* (2) *and have continuous second derivatives,* $0 \leqq x \leqq 1$, *then they are expansible in the form*

$$\begin{aligned} f(x) &\equiv \sum_{n=-\infty}^{+\infty} c_n u_n(x), \\ g(x) &= \sum_{n=-\infty}^{+\infty} c_n v_n(x), \end{aligned} \quad (96)$$

where

$$c_n \equiv \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx, \quad (97)$$

and (u_n, v_n) *represents the set of normal orthogonal solutions of* (1), (2).

In case λ_k *is a double value we count it as two and take any two linearly independent normal orthogonal solutions for that value of* λ.

Let us first show that two solutions (u_m, v_m), (u_n, v_n) of (1), (2) for different values of λ are orthogonal, i.e.,

$$\int_0^1 [u_m(s)u_n(s) + v_m(s)v_n(s)]ds = 0. \tag{98}$$

It is only necessary to apply Lemmas I, III, Section I, and divide by $\lambda_m - \lambda_n$. Obviously they are easily normalized by dividing in the case of (u_m, v_m) by $\sqrt{\int_0^1 (u_m^2 + v_m^2)dx}$.

If we assume the series of (96) to be uniformly convergent, then

$$\int_0^1 [f(x)u_m(x) + g(x)v_m(x)]dx = c_m$$

and the series become

$$\begin{cases} \sum_{n=-\infty}^{+\infty} u_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx, \\ \sum_{n=-\infty}^{+\infty} v_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx. \end{cases} \tag{99}$$

To study these we assume $|\lambda_n| > |a(x)|$, also $> |b(x)|$. We have

$$u_n'(x) - (\lambda_n + a)v_n(x) = 0,$$
$$v_n'(x) + (\lambda_n + b)u_n(x) = 0.$$

Then

$$\int_0^1 f(x)u_n(x)dx = -\int_0^1 \frac{f(x)}{\lambda_n + b} v_n'(x)dx$$
$$= \left[-\frac{f(x)v_n(x)}{\lambda_n + b} \right]_0^1 + \int_0^1 v_n(x) \frac{d}{dx}\left[\frac{f(x)}{\lambda_n + b(x)} \right] dx$$
$$= \left[-\frac{fv_n}{\lambda_n + b} \right]_0^1 + \int_0^1 \frac{u_n'(x)}{\lambda_n + a} \frac{d}{dx}\left[\frac{f}{\lambda_n + b} \right] dx.$$

But since by the Corollary of Theorem IV $O\left(\frac{1}{\lambda_n}\right) = O\left(\frac{1}{n\pi}\right) = O\left(\frac{1}{n}\right)$, for $|n|$ large, we have

$$\frac{d}{dx}\left(\frac{f(x)}{\lambda_n + b(x)} \right) \Big/ (\lambda_n + a(x)) = \frac{f'(x)}{\lambda_n + b} - \frac{f(x)b'(x)}{[\lambda_n + b]^2} \Big/ (\lambda_n + a) = O\left(\frac{1}{n^2}\right).$$

$$\therefore \quad \int_0^1 fu_n dx = \left[-\frac{fv_n}{\lambda_n + b} \right]_0^1 + O\left(\frac{1}{n^2}\right).$$

Similarly

$$\int_0^1 gv_n dx = \left[\frac{gu_n}{\lambda_n + a} \right]_0^1 + O\left(\frac{1}{n^2}\right).$$

Hence

$$c_n = \left[\frac{gu_n}{\lambda_n + a} - \frac{fv_n}{\lambda_n + b} \right]_0^1 + O\left(\frac{1}{n^2}\right)$$

or

$$c_n = \frac{1}{\lambda_n}[gu_n - fv_n]_0^1 + O\left(\frac{1}{n^2}\right). \tag{100}$$

Since by hypothesis f, g satisfy

$$U_1(f, g) = U_2(f, g) = 0, \tag{101}$$

by Lemma III, Section I, $c_n = O(1/n^2)$ and each series in (99) converges uniformly. If $f(x)$, $g(x)$ are expansible, we get the series (99) for them. To show that these series converge uniformly to $f(x)$, $g(x)$ respectively, we define

$$F(x) \equiv \sum_{n=-\infty}^{+\infty} u_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx,$$
$$G(x) \equiv \sum_{n=-\infty}^{+\infty} v_n(x) \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx.$$

Then formally we have

$$\int_0^1 [F(x)u_n(x) + G(x)v_n(x)]dx \equiv \int_0^1 [f(x)u_n(x) + g(x)v_n(x)]dx$$

or

$$\int_0^1 \{[F(x) - f(x)]u_n(x) + [G(x) - g(x)]v_n(x)\}dx = 0,$$
$$n = 0, \pm 1, \pm 2, \cdots.$$

If now we define

$$F(x) - f(x) \equiv \psi(x), \qquad G(x) - g(x) = -\varphi(x),$$

then by Theorem II we have

$$F(x) \equiv f(x), \qquad G(x) \equiv g(x), \qquad \text{q.e.d.}$$

THEOREM VII: *If $f(x)$, $g(x)$ do not satisfy* (2) *but possess continuous second derivatives as before, the series*

$$\begin{cases} c_0u_0(x) + [c_1u_1(x) + c_{-1}u_{-1}(x)] + [c_2u_2(x) + c_{-2}u_{-2}(x)] + \cdots, \\ c_0v_0(x) + [c_1v_1(x) + c_{-1}v_{-1}(x)] + [c_2v_2(x) + c_{-2}v_{-2}(x)] + \cdots, \end{cases} \tag{102}$$

converge uniformly to $f(x)$, $g(x)$ for $0 < \epsilon \leqq x \leqq 1 - \epsilon < 1$, where ϵ is an arbitrarily small positive number and (u_n, v_n), c_n are defined as before.

We have from (100)

$$c_nu_n(x) = \frac{u_n(x)}{\lambda_n}[g(x)u_n(x) - f(x)v_n(x)]_0^1 + O\left(\frac{1}{n^2}\right).$$

Hence by (91) for $|n|$ large

$$c_nu_n(x) = \frac{(-1)^nK_1u_n(x) - K_0u_n(x)}{n\pi} + O\left(\frac{1}{n^2}\right),$$

where

$$K_1 \equiv g(1)\sin P(1) - f(1)\cos P(1),$$
$$K_0 \equiv g(0)\sin P(0) - f(0)\cos P(0).$$

Again

$$c_{-n}u_{-n}(x) = \frac{(-1)^n K_1 u_{-n}(x) - K_0 u_{-n}(x)}{-n\pi} + O\left(\frac{1}{n^2}\right).$$

Hence

$$c_n u_n(x) + c_{-n}u_{-n}(x) = \left[\frac{(-1)^n K_1}{n\pi} - \frac{K_0}{n\pi}\right][u_n - u_{-n}].$$

But

$$u_n(x) - u_{-n}(x) = \sin[n\pi x + P(x)] - \sin[-n\pi x + P(x)] + O\left(\frac{1}{n}\right)$$
$$= 2\sin(n\pi x)\cos P(x) + O\left(\frac{1}{n}\right).$$

$$\therefore\ c_n u_n(x) + c_{-n}u_{-n}(x)$$
$$= 2\cos P(x)\left[K_1(-1)^n \frac{\sin n\pi x}{n\pi} - K_0 \frac{\sin n\pi x}{n\pi}\right] + O\left(\frac{1}{n^2}\right).$$

The series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi x}{n\pi}$, $\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n\pi}$ converge uniformly, for $0 < \epsilon \leqq x \leqq 1 - \epsilon < 1$, and since we get a similar expression for

$$c_n v_n(x) + c_{-n}v_{-n}(x),$$

the rest of the proof is like in the Theorem above.

Section V. Further Results on the Distribution of Principal Parameter Values.

Without the use of Theorem IV we can show that for the case in which

$$|C| < \sqrt{A^2 + B^2},$$

$D(\lambda)$ has exactly two roots in each interval

$$[(2k-1)\pi, (2k+1)\pi], \quad |k| \text{ large and integral,}$$

which have the asymptotic form (95) for $|n|$, $|\lambda|$ sufficiently large. Without loss of generality assume (58), then by definition $C = 2$. Then

$$\sqrt{A^2 + B^2} - 2 \equiv E > 0. \tag{103}$$

By (50)

$$D(\lambda) = \sqrt{A^2 + B^2}\sin[\xi(1) + \varphi] + C + \frac{K}{\lambda}, \qquad 0 \leqq |K| < M,$$

for $|\lambda|$ large, or

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = \sin\Lambda + \frac{C}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda},$$

where

$$\Lambda \equiv \xi(1) + \varphi. \tag{104}$$

For $\Lambda = (2k-1)\pi$ or $2k\pi$, and $|\lambda| > \lambda_M$,

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = \frac{2}{\sqrt{A^2 + B^2}} + \frac{K'}{\lambda} > 0.$$

For $\Lambda = (4k-1)\pi/2$, and $|\lambda| > \lambda_E$,

$$\frac{D(\lambda)}{\sqrt{A^2+B^2}} = -1 + \frac{2}{\sqrt{A^2+B^2}} + \frac{K'}{\lambda} = \frac{-E}{\sqrt{A^2+B^2}} + \frac{K'}{\lambda} < 0.$$

Let L_1 be the larger of λ_M, λ_E, then for $|\lambda| < L_1$, $D(\lambda)$ must have at least two roots in the intervals $(2k-1)\pi < \lambda < 2k\pi$. Since for $2k\pi \leqq \Lambda \leqq (2k+1)\pi$, $\sin\Lambda \geqq 0$, $D(\lambda)$ has no root there. To show that there are exactly two roots in the previous interval we use the fact that $D(\lambda) = 0$ only when

$$\sin\Lambda = -\frac{2}{\sqrt{A^2+B^2}} + \frac{K'}{\lambda}.$$

It is obvious that for $|\lambda| >$ some L_2 the right member is always negative since $|K'| < M$. If L is the larger of L_1, L_2, then, since

$$\sin\Lambda = \frac{E}{\sqrt{A^2+B^2}} - 1 + \frac{K'}{\lambda},$$

and when $|\lambda| > \lambda_E$

$$\frac{K'}{\lambda} < \frac{E}{\sqrt{A^2+B^2}},$$

we have

$$0 > \sin\Lambda > -1, \qquad |\lambda| > L.$$

Obviously there are just two values of Λ in any $[(2k-1)\pi,\ 2k\pi]$ interval which satisfy this inequality, i.e.,

$$\Lambda_1 = (2k-1)\pi + \psi + O\left(\frac{1}{\lambda}\right),$$

$$\Lambda_2 = 2k\pi - \psi + O\left(\frac{1}{\lambda}\right), \qquad 0 < \psi < \frac{\pi}{2},$$

where $\sin\psi \equiv \dfrac{-2}{\sqrt{A^2+B^2}}$.

$$\therefore\ \Lambda = m\pi + (-1)^{m+1}\psi + O\left(\frac{1}{\lambda}\right),$$

for $|m|$ large, and by the definition of Λ

$$\lambda_n = m\pi + (-1)^{m+1}\psi - \int_0^1 \frac{a+b}{2}\,dx - \varphi + O\left(\frac{1}{m}\right),$$

when $|n|$ is large; or since in each 2π-interval there are 2 values of λ we have

$$\lambda_n = n\pi + s\pi + (-1)^{n+s+1}\psi - \int_0^1 \frac{a+b}{2}\,dx - \varphi + O\left(\frac{1}{n}\right),$$

s being a finite integer.

CASE of $C^2 = A^2 + B^2$: Here the above reasoning does not apply and we attack the problem by getting a more accurate asymptotic expansion

for $D(\lambda)$ and $D'(\lambda)$. If we assume

$$u(x) \equiv e^{\lambda\omega}\left[u_0 + \frac{u_1}{\lambda} + \frac{u_2}{\lambda^2} + \cdots\right],$$
$$v(x) \equiv e^{\lambda\omega}\left[v_0 + \frac{v_1}{\lambda} + \frac{v_2}{\lambda^2} + \cdots\right],$$

then substituting in (1) equate coefficients, we are led to the expansion

$$u(x) \equiv A_0 e^{\xi i} + B_0 e^{-\xi i} + \sum_{k=1}^{\infty} \frac{A_k e^{\xi i} + B_k e^{-\xi i}}{\lambda^k},$$
$$v(x) \equiv C_0 e^{\xi i} + D_0 e^{-\xi i} + \sum_{k=1}^{\infty} \frac{C_k e^{\xi i} + D_k e^{-\xi i}}{\lambda^k},$$

or since we restrict λ to real values, to

$$\begin{cases} u(x) \equiv A_0 \cos\xi + B_0 \sin\xi + \sum_{k=1}^{\infty} \dfrac{A_k \cos\xi + B_k \sin\xi}{\lambda^k}, \\ v(x) \equiv C_0 \cos\xi + D_0 \sin\xi + \sum_{k=1}^{\infty} \dfrac{C_k \cos\xi + D_k \sin\xi}{\lambda^k}, \end{cases} \tag{105}$$

where ξ is defined as in (44) and A, B, C, D are independent of λ.

We may show that $(\alpha\gamma) = (\beta\delta)$, $(\alpha\delta) = (\gamma\beta)$ so that

$$A = 2(\alpha\gamma), \qquad B = 2(\alpha\delta), \qquad C = 2(\alpha\beta),$$

and since $(\alpha\beta) = 1$, we have

$$\frac{D(\lambda)}{\sqrt{A^2 + B^2}} = 1 + [u_2(1) - v_1(1)]^{\frac{1}{2}} \cos\varphi + [u_1(1) + v_2(1)]^{\frac{1}{2}} \sin\varphi,$$

in which (u_1, v_1), (u_2, v_2) are defined by (12) and φ by (52). From (50) we can show that $D'(\lambda)$ has a root in the interval

$$(2k-1)\pi + \tfrac{1}{4}\pi < \Lambda < (2k-1)\pi + \tfrac{3}{4}\pi. \tag{106}$$

By putting (105) in (1) and equating coefficients we get a set of equations from which we can solve for A, B, C, D and if we put that value of λ in for which $D'(\lambda) = 0$, we obtain

$$D(\lambda) - \frac{m_1 + 2m_0 m_1 \cos 2\varphi + m_0^2}{8\lambda^2} + O\left(\frac{1}{\lambda^3}\right),$$

wherein

$$m_1 \equiv \tfrac{1}{2}a(1) - \tfrac{1}{2}b(1), \qquad m_0 \equiv \tfrac{1}{2}a(0) - \tfrac{1}{2}b(0).$$

Hence $D(\lambda)$ will have two roots in the interval (106) unless

Case 1) $m_0 = m_1 = 0$;
Case 2) $\varphi \equiv \cos^{-1}(\alpha\gamma) \equiv \sin^{-1}(\alpha\delta) = \frac{1}{2}\pi$, $m_0 = m_1$;
Case 3) $\varphi = 0$, $m_0 = -m_1$.

By extending the calculation to the fifth set of functions A_4, B_4, C_4, D_4 we find that $D(\lambda)$ will still have two roots in (106) for Case 1) unless

$$m_1'^2 + 2m_0'm_1' \cos 2\varphi + m_0'^2 = 0,$$

i.e., unless

$$\begin{aligned} &1°. \quad m_0' = m_1' = 0; \\ &2°. \quad \varphi = \tfrac{1}{2}\pi, \quad m_0' = m_1'; \\ &3°. \quad \varphi = 0, \quad m_0' = -m_1'; \end{aligned}$$

in which

$$m_0' \equiv \tfrac{1}{2}a'(0) - \tfrac{1}{2}b'(0), \qquad m_1' \equiv \tfrac{1}{2}a'(1) - \tfrac{1}{2}b'(1).$$

The same is true for Case 2) unless

$$(m_1' - m_0')^2 + 4m_0^2(l_1 - l_0)^2 = 0$$

or

$$\begin{aligned} &1°. \quad m_1' = m_0', \quad m_0 = 0; \\ &2°. \quad m_1' = m_0', \quad l_1 = l_0; \end{aligned}$$

where

$$l_1 \equiv \tfrac{1}{2}a(1) + \tfrac{1}{2}b(1), \qquad l_0 \equiv \tfrac{1}{2}a(0) + \tfrac{1}{2}b(0).$$

It is probable that by this method $D(\lambda)$ would also be shown to have roots for Case 3) unless

$$(m_1' + m_0')^2 + 4m_0^2(l_1 + l_0)^2 = 0,$$

and that if one should continue the computation one would find that $D(\lambda)$ possesses roots unless

$$\begin{aligned} &\text{I.} \quad m_0 = m_1 = m_1' = m_0' = m_1'' = m_0'' \cdots = 0; \\ &\text{II.} \quad \varphi = \tfrac{1}{2}\pi, \quad m_0 = m_1, \quad m_0' = m_1', \quad m_0'' = m_1'', \quad \cdots; \\ &\text{III.} \quad \varphi = 0, \quad m_0 = -m_1, \quad m_0' = -m_1', \quad m_0'' = -m_1'', \quad \cdots. \end{aligned}$$

It is interesting to note that for a restricted case of I, namely

$$a(x) \equiv b(x) \equiv 0,$$

$D(\lambda)$ possesses only double roots, provided

$$\begin{cases} \alpha_1 = \gamma_1 = \beta_2 = \delta_2 = 1, \\ \alpha_2 = \gamma_2 = \beta_1 = \delta_1 = 0. \end{cases} \tag{107}$$

Here

$$\begin{aligned} D(\lambda) &= 2 + 2\cos\lambda, \\ \lambda &= (2k-1)\pi, \end{aligned}$$

in fact

$$D(\lambda) \equiv \begin{vmatrix} 1 + \cos\lambda, & \sin\lambda \\ -\sin\lambda, & 1 + \cos\lambda \end{vmatrix}.$$

This leads to a special case of Fourier's Series in which the terms involving $\sin n\pi x$, $\cos n\pi x$ for n even are wanting.

Another interesting result is the following

THEOREM: *A necessary condition that the determinant*

$$D(\lambda) \equiv \begin{vmatrix} U_1(u_1v_1) & U_1(u_2v_2) \\ U_2(u_1v_1) & U_2(u_2v_2) \end{vmatrix}$$

be of rank zero is that $(\alpha\beta) = (\gamma\delta)$, *where* (u_1, v_1), (u_2, v_2) *are defined by* (12).

We have

$$U_1(u_1v_1) \equiv \alpha_1 + \gamma_1 u_1(1) + \delta_1 v_1(1) = 0, \tag{108}$$

$$U_2(u_1v_1) \equiv \alpha_2 + \gamma_2 u_1(1) + \delta_2 v_1(1) = 0, \tag{109}$$

$$U_1(u_2v_2) \equiv \beta_1 + \gamma_1 u_2(1) + \delta_1 v_2(1) = 0, \tag{110}$$

$$U_2(u_2v_2) \equiv \beta_2 + \gamma_2 u_2(1) + \delta_2 v_2(1) = 0, \tag{111}$$

Multiply the first two equations by δ_2, $-\delta_1$ respectively, then

$$(\alpha\delta) + (\gamma\delta)u_1(1) = 0. \tag{112}$$

Similarly using $-\alpha_2$, α_1 on the last two we have

$$(\alpha\beta) + (\alpha\gamma)u_2(1) + (\alpha\delta)v_2(1) = 0. \tag{113}$$

Again using γ_2, $-\gamma_1$ to multiply the first two, we get

$$(\alpha\gamma) - (\gamma\delta)v_1(1) = 0. \tag{114}$$

Combining equations (112), (114), (113) we obtain

$$(\alpha\beta) - (\gamma\delta)\begin{vmatrix} u_1(1), & v_1(1) \\ u_2(1), & v_2(1) \end{vmatrix} = 0,$$

or

$$(\alpha\beta) = (\gamma\delta).$$ Q.e.d.

CONFORMAL TRANSFORMATIONS OF PERIOD n AND GROUPS GENERATED BY THEM.*

By Harry Langman.

Introduction.

Professor Kasner† has discussed the characteristics of groups of transformations generated by conformal transformations of period 2. He considered both the "direct" and what he terms "reverse," or "improper" transformations. The former type may be represented in the form $Z = f(z)$, where $f(z)$ is analytic at the origin, and converts it into itself. The latter type, termed a "symmetry," may be represented in the form $Z = f(z_0)$, where z_0 is the conjugate of z.

In discussing direct transformations, Professor Kasner utilizes the implicit form

$$Z + z = d_2(Z - z)^2 + d_4(Z - z)^4 + \cdots,$$

which includes every transformation of period 2, and every solution of which, for arbitrary values of the coefficients d, yields a transformation of period 2. In considering the conditions under which a given transformation

$$Z = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

can be factored as the product of two transformations of period 2, Kasner obtains the condition $c_3 - c_2^2 = 0$, besides the obvious condition $c_1 = 1$. If $c_2 = 0$, further conditions must be satisfied.

In the case of reverse transformations, the given transformation can always be factored into two "symmetries" if $|c_1| = 1$, and the angle of c_1 is incommensurable with π. If the latter condition is not satisfied, then the given transformation can in any case be factored into four symmetries.

It is the purpose of this paper to generalize the results obtained by Kasner to include transformations of period n. In the case of direct transformations, it is remarkable that no such necessary condition as that obtained by Kasner upon the coefficients following the first is found necessary for factorization into transformations of periods greater than 2. In the case $c_1 = 1$ there is, however, a non-zero relation between the coefficients immediately following the first in the given transformation and the period of the factor transformations (the periods of the latter then being neces-

* Presented to the American Mathematical Society December 28, 1920.

† "Infinite Groups Generated by Conformal Transformations of Period 2 (Involutions and Symmetries)," American Journal of Mathematics, XXXVIII, 2, 1916.

sarily equal). In any case, this condition can be obviated by either changing the period of the factor transformations or factoring into three transformations of given period. The complete result is given in Theorem VI.

In the case of reverse conformal transformations, it will be shown that all such transformations are of irreducible period 2; therefore no others exist than those discussed by Kasner.

The parametric form utilized for transformations of regular period is a general solution of Babbage's equation,* and the reduction to the form $f = \varphi^{-1}\epsilon\varphi$ yields $\varphi f = \epsilon\varphi$, a special case of Schröder's equation.†

In the following discussion, the terms function and transformation are used interchangeably, and questions of convergence are not gone into, the analysis being purely formal.

PRELIMINARY DISCUSSION; PARAMETRIC FORM OF PERIODIC TRANSFORMATIONS.

1. Let $f(z)$ be defined by

$$f(z) = \lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \tag{1}$$

where $\lambda_1 \neq 0$. We may introduce the notation

$$f_1(z) = f(z), \qquad f_{s+1}(z) = f[f_s(z)]; \qquad s = 1, 2, 3, \cdots. \tag{2}$$

Assuming the transformation defined by (1) to be of period n, we have

$$f_n(z) = z, \qquad f_{kn+s}(z) = f_s(z); \qquad k = 1, 2, 3, \cdots. \tag{3}$$

We have obviously

$$\lambda_1^n = 1. \tag{4}$$

If in (1) we choose $\lambda_1 = 1$, we have

$$f_n(z) = z + n\lambda_2 z^2 + \cdots;$$

hence we must have $\lambda_2 = 0$. Similarly, all the other coefficients vanish. Hence we have

THEOREM I. *If the transformation*

$$\lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots$$

be of period n, we must have $\lambda_1^n = 1$; *if* $\lambda_1 = 1$, *the transformation reduces to identity.*

* S. Pincherle: "Functional Equations and Operations," *Encyklopädie d. Math. Wiss.*, II, A 11, and *Encyclopedie d. Sci. Math.*, II, 26. Also O. Rausenberger, "Lehrbuch der Theorie der periodischen Funktionen," Leipzig, 1884, p. 162; A. A. Bennett, "The Iteration of Functions of One Variable," *Annals of Math.*, 2d series, 17 (1915).

† Ibid. In our case ϵ is taken so that $\epsilon^n = 1$. G. A. Pfeiffer, *Trans. Am. Math. Soc.*, XVIII, 2, pp. 185–198, considers the complementary case, $|\epsilon| = 1$ with incommensurable angle, and discusses the convergence and divergence of the solutions obtained.

2. Assuming (1) to be of period n, we may write it in the form

$$f(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots; \qquad \epsilon^n = 1. \tag{5}$$

Suppose now ϵ *not* a primitive root of $\epsilon^n = 1$. Suppose ϵ a primitive root of $\epsilon^m = 1$. Then $rm = n$, where $r > 1$. Denoting $f_m(z)$ by $g(z)$, we have

$$g_r(z) = f_{rm}(z) = z.$$

We have obviously $g(z) = \epsilon^m z + \cdots$; hence, by Theorem I,

$$g(z) = f_m(z) = z.$$

Hence the transformation (5) is of period m. We have then

THEOREM II. *If a periodic transformation be expressed in the form*

$$\epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots,$$

where

$$\epsilon^m = 1,$$

*then the transformation is of period m.**

Hence in the form (5) we may conveniently restrict ourselves to the case where ϵ is a primitive root of $\epsilon^n = 1$. In the following discussion, we shall presume this to be the case.

3. Suppose we consider the n variables $z_1, z_2, \cdots, z_n$, between which we have the $n - 1$ relations

$$z_{t+1} = f(z_t); \qquad t = 1, 2, \cdots, n - 1, \tag{6}$$

where

$$f(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \tag{7}$$

ϵ being a primitive nth root of unity. We may introduce the linear substitution

$$z_t = \epsilon^t x_1 + \epsilon^{2t} x_2 + \cdots + \epsilon^{st} x_s + \cdots + \epsilon^{nt} x_n; \quad t = 1, 2, \cdots, n. \tag{8}$$

This is reversible; we have

$$x_s = \frac{1}{n}(\epsilon^{-s} z_1 + \epsilon^{-2s} z_2 + \cdots + \epsilon^{-ts} z_t + \cdots + \epsilon^{-ns} z_n); \quad s = 1, 2, \cdots, n. \tag{9}$$

By means of (6) and (7), each variable z can be expressed formally as a power series in z_1. We have

$$z_t = \epsilon^{t-1} z_1 + \cdots; \qquad t = 2, 3, \cdots, n. \tag{10}$$

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* If (5) take the form

$$f(z) = \epsilon z + \lambda_r z^r + \cdots,$$

we obtain

$$f_n(z) = \epsilon^n z + \epsilon^{n-1}\lambda_r[1 + \epsilon^{r-1} + \epsilon^{2(r-1)} + \cdots + \epsilon^{(n-1)(r-1)}]z^r + \cdots.$$

Hence if $r - 1 \equiv 0 \pmod{n}$, we must have $\lambda_r = 0$. Hence the next coefficient to appear after the first cannot be of order $kn + 1$.

From (9), then,

$$x_1 = \epsilon^{-1}z_1 + \cdots. \tag{11}$$

From (11), we have z_1 expressible as a power series in x_1. Hence from (10) each z is so expressible. From (9), then, each x can be expanded formally as a power series in x_1 with coefficients uniquely determined in terms of those of f in (7). We may then write

$$x_s = A_{s,1}x_1 + A_{s,2}x_1^2 + \cdots + A_{s,r}x_1^r + \cdots; \quad s = 2, 3, \cdots, n. \tag{12}$$

Hence the $n - 1$ relations (6) among the variables z may be replaced by the $n - 1$ implicit relations (12) where each variable x is expressed in terms of the variables z in the form (9). If z_2 is defined in terms of z_1 in the form (7), then we may introduce the additional variables $z_3, z_4, \cdots, z_n$, similarly defined; hence z_2 may be defined in terms of z_1 in the implicit form (12), involving the elimination of the additional variables $z_3, z_4, \cdots, z_n$.

4. Suppose now the transformation defined by (7) to be of period n. Then, since $z_t = f_{t-1}(z_1)$,

$$z_{n+1} = f_n(z_1) = z_1 = f(z_n). \tag{13}$$

Using the substitution (8), we have then the following n relations between the variables x:

$$\begin{aligned}
&\epsilon^2x_1 + \cdots + \epsilon^{2s}x_s + \cdots + \epsilon^{2n}x_n \\
&\qquad = f(\epsilon x_1 + \cdots + \epsilon^s x_s + \cdots + \epsilon^n x_n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
&\epsilon^{t+1}x_1 + \cdots + \epsilon^{s(t+1)}x_s + \cdots + \epsilon^{n(t+1)}x_n \\
&\qquad = f(\epsilon^t x_1 + \cdots + \epsilon^{st}x_s + \cdots + \epsilon^{nt}x_n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
&\epsilon^n x_1 + \cdots + \epsilon^{sn}x_s + \cdots + \epsilon^{nn}x_n \\
&\qquad = f(\epsilon^{n-1}x_1 + \cdots + \epsilon^{s(n-1)}x_s + \cdots + \epsilon^{n(n-1)}x_n), \\
&\epsilon x_1 + \cdots + \epsilon^s x_s + \cdots + \epsilon^n x_n \\
&\qquad = f(\epsilon^n x_1 + \cdots + \epsilon^{sn}x_s + \cdots + \epsilon^{nn}x_n).
\end{aligned} \tag{14}$$

From the previous discussion, we note that the first $n - 1$ of equations (14) will yield unique expansions in power series of x_1 in the form (12) for the variables $x_2, x_3, \cdots, x_n$. We may represent the relations (12) in the following notation:

$$x_s = \theta_s(x_1); \qquad s = 2, 3, \cdots, n. \tag{15}$$

If we now introduce in (14) the substitution

$$x_s = \epsilon^s y_s; \qquad s = 1, 2, \cdots, n, \tag{16}$$

equations (15) become

$$y_s = \epsilon^{-s}\theta_s(\epsilon y_1); \qquad s = 2, 3, \cdots, n. \tag{17}$$

We should observe, however, that the last $n-1$ equations of (14) also yield definite power series in x_1 for the other variables x and that these must be identical with the relations (15). But on introducing the quantities y in the first $n-1$ equations of (14), in the form (16), we observe that the relations between the variables y are identical with those between the corresponding variables x in the last $n-1$ equations of (14). Hence the relations between the variables y must be the same as those between the corresponding variables x. Hence we have

$$y_s = \theta_s(y_1); \qquad s = 2, 3, \cdots, n. \tag{18}$$

These of course must be consistent with (17). Hence

$$\theta_s(y_1) = \epsilon^{-s}\theta_s(\epsilon y_1); \qquad s = 2, 3, \cdots, n. \tag{19}$$

Replacing y_1 in (19) successively by ϵy_1, we obtain, on dropping the subscript,

$$\theta_s(y) = \epsilon^{-ts}\theta_s(\epsilon^t y); \qquad s = 2, 3, \cdots; \qquad t = 1, 2, 3, \cdots. \tag{20}$$

From the form (12), (20) may be written

$$\theta_s(y) = \epsilon^{-ts}\sum_{r=1}^{\infty} \epsilon^{tr} A_{s,\,r}\cdot y^r; \quad s = 2, 3, \cdots, n; \quad t = 1, 2, 3, \cdots. \tag{21}$$

Letting t assume the range of values* 1 to n in (21), and adding the resulting series, we obtain the form

$$\theta_s(y) = A_{s,\,s}y^s + A_{s,\,s+n}y^{s+n} + A_{s,\,s+2n}y^{s+2n} + \cdots; \quad s = 2, 3, \cdots, n. \tag{22}$$

From the form of the series (22) we observe that the double subscripts are not now necessary. We may then write the series (15) in the form

$$\begin{aligned}
x_2 &= a_2x_1^2 + a_{n+2}x_1^{n+2} + a_{2n+2}x_1^{2n+2} + \cdots,\\
x_3 &= a_3x_1^3 + a_{n+3}x_1^{n+3} + a_{2n+3}x_1^{2n+3} + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots,\\
x_s &= a_sx_1^s + a_{n+s}x_1^{n+s} + a_{2n+s}x_1^{2n+s} + \cdots,\\
&\cdots\cdots\cdots\cdots\cdots,\\
x_n &= a_nx_1^n + a_{2n}x_1^{2n} + a_{3n}x_1^{3n} + \cdots.
\end{aligned} \tag{23}$$

Replacing the variables x in (23)† by the corresponding expressions in the variables z from (9), we have then $n-1$ implicit relations among those variables in place of the $n-1$ direct relations (6).

5. If we now write

$$\frac{1}{n}(\epsilon^{-1}z_1 + \epsilon^{-2}z_2 + \cdots + \epsilon^{-t}z_t + \cdots + \epsilon^{-n}z_n) = r, \tag{24}$$

* This is equivalent to permuting the substitution (16) in equations (14).

† These equations have already been obtained by Bennett, loc. cit.

equations (23) may be written

$$\begin{aligned}
&\frac{1}{n}(\epsilon^{-2}z_1+\epsilon^{-4}z_2+\cdots+\epsilon^{-2t}z_t+\cdots+\epsilon^{-2n}z_n)=a_2r^2+a_{n+2}r^{n+2}+\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\frac{1}{n}(\epsilon^{-s}z_1+\epsilon^{-2s}z_2+\cdots+\epsilon^{-ts}z_t+\cdots+\epsilon^{-ns}z_n)=a_sr^s+a_{n+s}r^{n+s}+\cdots, \qquad (25)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\frac{1}{n}(\epsilon^{-n}z_1+\epsilon^{-2n}z_2+\cdots+\epsilon^{-tn}z_t+\cdots+\epsilon^{-nn}z_n)=a_nr^n+a_{2n}r^{2n}+\cdots.
\end{aligned}$$

Multiplying equations (24) and (25) seriatim by ϵ^t, ϵ^{2t}, $\cdots$, ϵ^{st}, $\cdots$, ϵ^{nt}, and adding, we obtain

$$\begin{aligned}
z_t = {} & \epsilon^t r + \epsilon^{2t}(a_2r^2 + a_{n+2}r^{n+2} + \cdots)\\
& + \cdots\cdots\cdots\cdots\\
& + \epsilon^{st}(a_sr^s + a_{n+s}r^{n+s} + \cdots) \qquad (26)\\
& + \cdots\cdots\cdots\cdots\\
& + \epsilon^{nt}(a_nr^n + a_{2n}r^{2n} + \cdots),
\end{aligned}$$

for all values of t from 1 to n. If we now define the function φ by

$$\varphi(z) = z + \sum_{k=0}^{\infty}\sum_{r=1}^{n-1} a_{kn+r+1}z^{kn+r+1}, \qquad (27)$$

equations (26) take the form

$$z_t = \varphi(\epsilon^t r); \qquad t = 1, 2, \cdots, n. \qquad (28)$$

In particular,

$$z_1 = \varphi(\epsilon r), \qquad z_2 = \varphi(\epsilon^2 r), \qquad (29)$$

which are equivalent to

$$z_1 = \varphi(r), \qquad z_2 = \varphi(\epsilon r), \qquad (29')$$

defining z_2 as a function of z_1 in terms of the parameter r. Furthermore, every pair of equations (28) determines either z uniquely as an integral power series in the other with coefficients expressible as polynomials in the coefficients a in (27) of the same and lower orders. This is obviously also true if instead of φ as defined by (27) we use the more general form

$$\psi(r) = r + \sum_{t=2}^{\infty} a_t r^t. \qquad (30)$$

6. From the form of equations (28) we observe that if for an arbitrary set of coefficients a we obtain $z_2 = f(z_1)$, we must also have $z_3 = f(z_2)$, etc. Hence $z_{n+1} = f(z_n)$. But $z_1 = z_{n+1} = z_{2n+1}$, etc. Hence $z_1 = f(z_n)$, from which $f_n(z) = z$. Hence $f(z)$ is of period n. Furthermore, if z_α and z_β be any two of the z's defined by (28), either is expressible as a periodic

function of the other. If $|\alpha - \beta|$ is prime to n, the order is also n. If only equations (29) are given, or the corresponding equations with ψ instead of φ, the remaining z's may be introduced in the form (28), yielding z_2 as a function of z_1 of period n. Hence we have

THEOREM III. *Every transformation of period n, $y = f(z)$, can be put uniquely into the form*

$$z = \varphi(r), \qquad y = \varphi(\epsilon r); \qquad \epsilon^n = 1,$$

where

$$\varphi(r) = r + \sum_{k=0}^{\infty} \sum_{t=1}^{n-1} a_{kn+t+1} r^{kn+t+1};$$

and every solution of these equations for arbitrary values of the coefficients defines $y = f(z)$ uniquely as a transformation of period n.

*In other words, a necessary and sufficient condition for $y = f(z)$ to be a transformation of period n is that it be a solution of equations of the form $z = \varphi(r)$, $y = \varphi(\epsilon r)$.**

Hence the two forms are equivalent, and we may conveniently confine our attention to the inclusive form (29′).

7. If the transformation (5) result from the elimination of r between the equations (29′), then we obtain

$$\lambda_t = (\epsilon^t - \epsilon)a_t + (\text{polynomial in } \lambda\text{'s and } a\text{'s of order} < t),$$

which can obviously be put into the form

$$\lambda_t = (\epsilon^t - \epsilon)a_t + (\text{polynomial in } a\text{'s of order} < t). \tag{31}$$

The quantities a in (31) are arbitrary. For each new λ_t, where t is not of the form $kn + 1$, a new arbitrary a_t is introduced. Furthermore, each a can be expressed as a polynomial in λ's of corresponding and lower orders. Hence the λ's in (5) of orders not of the form $kn + 1$ can be taken arbitrarily, the remaining λ's being then determined necessarily in the form

$$\lambda_{kn+1} = R_k(\lambda_{kn}, \lambda_{kn-1}, \cdots); \qquad k = 1, 2, \cdots, \tag{32}$$

where the R's are rational integral functions. Hence we have

THEOREM IV. *If*

$$y = \epsilon z + \lambda_2 z^2 + \cdots; \qquad \epsilon^n = 1$$

be a transformation of period n, all coefficients of orders not of the form $kn + 1$

* If $n = 2$, $\epsilon = -1$, and equations (27) and (29′) yield $z_1 = r + a_2 r^2 + a_4 r^4 + \cdots$, $z_2 = -r + a_2 r^2 + a_4 r^4 + \cdots$, from which

$$\frac{z_2 + z_1}{2} = a_2 \left(\frac{z_2 - z_1}{2}\right)^2 + a_4 \left(\frac{z_2 - z_1}{2}\right)^4 + \cdots,$$

the form utilized by Kasner.

may be taken arbitrarily, the remaining coefficients being determined as necessary rational integral functions of the coefficients of lower order.

In other words, $k(n-1)$ out of the kn coefficients following the first may be taken arbitrarily.

The relations (32) are then necessary and sufficient for the transformation (5) to be of period n. If now, instead of (29′), we define the relation $y = f(z)$ in the form

$$z = \psi(r), \qquad y = \psi(\epsilon r), \tag{33}$$

where ψ has the form (30), then f is also of period n. In other words, (33) is no more general than (29′). In this case too the relations (31) will not yield expressions in λ's for the a's of order $kn+1$. On the other hand, the λ's of order $kn+1$ will be expressed in terms of a's of order $k'n+1$, $k' < k$, as well as the other a's. But since the relations (32) between the λ's are *necessary*, it follows that the a's of order $k'n+1$ will be automatically eliminated on eliminating the remaining a's from the expressions for λ_{kn+1} and the λ's of lower order.

Hence, as already observed, any two equations $z_\alpha = \varphi(\epsilon^\alpha r)$ and $z_\beta = \varphi(\epsilon^\beta r)$, of (28), or the corresponding equations with ψ instead of φ, determine either z as a periodic function of the other. The period is the quotient between n and the greatest common divisor of n and $|\alpha - \beta|$.

8. If the function

$$f(z) = \lambda_1 z + \lambda_2 z^2 + \cdots \tag{34}$$

result from the relation

$$f(z) = g^{-1}[\epsilon g(z)]; \qquad \epsilon^n = 1, \tag{35}$$

where

$$g(z) = A_1 z + A_2 z^2 + \cdots; \qquad A_1 \neq 0, \tag{36}$$

then $f(z)$ is obviously of period n. We shall now show that every periodic transformation $f(z)$ can be expressed in the form (35).*

Suppose f is given where, symbolically, $f^n = 1$. We inquire now whether g can always be found so as to satisfy (35). This condition is equivalent to

$$g[f(z)] = \epsilon g(z). \tag{37}$$

It is to be observed that, whether we obtain f when g is given, or g when f is given, the *same* set of formal identities obtained by equating the coefficients of like powers of z in (37) must be considered. These take the form

$$\begin{aligned} A_1\lambda_1 &= \epsilon A_1, \qquad A_1\lambda_2 + A_2\epsilon^2 = \epsilon A_2, \qquad \cdots, \\ A_1\lambda_t + 2A_2\epsilon\lambda_{t-1} + \cdots + A_t\epsilon^t &= \epsilon A_t, \qquad \cdots. \end{aligned} \tag{38}$$

We observe that A_1 may be taken arbitrarily subject, of course, to the

* Cf. S. Pincherle, O. Rausenberger and A. A. Bennett, loc. cit.

condition $A_1 \neq 0$. We have $\lambda_1 = \epsilon$. Furthermore, each coefficient A_t, $t < n+1$, is expressed as A_1 (polynomial in λ's). Hence we have λ_{n+1} expressed as a necessary rational integral function of λ's of lower order. Similarly, λ's of order $kn+1$ are expressed as rational integral functions of lower λ's and of arbitrary A's of order $k'n+1$, $k' < k$. But since every solution $f(z)$ where $g(z)$ in (35) is given is of period n, we have then a set of *necessary* conditions for the coefficients λ of order $kn+1$ of the form (32). Hence on eliminating the coefficients A of orders other than those of the form $kn+1$ from the expressions for λ_{kn+1} and λ's of lower order, the coefficients $A_{k'n+1}$ $(k' < k)$ must be eliminated automatically, the resulting conditions being identical with those represented by (32).

Furthermore, if (34) is given, we can find a function $g(z)$ consistent with (35) only if the same set of conditions (32) are satisfied between the coefficients λ. But as just seen these conditions are none other than those necessary for $f(z)$ to be a periodic transformation of order n. Hence every transformation (34) of order n can be expressed in the form (35). Furthermore, given g, f results uniquely, but not vice versa, since the coefficients of form A_{kn+1} may be taken arbitrarily, subject to the single restriction $A_1 \neq 0$. Hence we may conveniently choose

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, \cdots,$$

from which g takes the form

$$g(z) = z + \sum_{k=0}^{\infty} \sum_{r=1}^{n-1} A_{kn+r+1} z^{kn+r+1}, \tag{39}$$

in which case, from the form of equations (38), each coefficient A is determined uniquely in terms of the coefficients λ of the given transformation (34). Hence we have

THEOREM V. *Every transformation of the form*

$$f(z) = g^{-1}[\epsilon g(z)], \qquad \epsilon^n = 1,$$

where

$$g(z) = A_1 z + A_2 z^2 + \cdots; \qquad A_1 \neq 0,$$

is of period n, and every transformation $f(z)$ of period n may be expressed in the form $g^{-1}[\epsilon g(z)]$; furthermore, if we choose

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, \cdots,$$

the function $g(z)$ corresponds uniquely to $f(z)$.

In other words, every periodic transformation is conformally reducible to the form $Y = \epsilon Z$, a rotation about the origin through the angle $2\pi/n$.*

It is interesting to compare the functions φ and g in (27) and (39). Obviously each corresponds uniquely to the other and either may be used in place of the other or its inverse in the above equations.

* A simple example of a periodic transformation is obtained by taking $g(z) = z/(z-1)$, from which $f(z) = \epsilon z/[(\epsilon - 1)z + 1]$.

9. The method followed in the preceding discussion will enable us to write down readily the general solutions of some special additional functional equations. Suppose f given, where $f^n = 1$, and it is required to find g where, symbolically, $g^m = f$.* We may anticipate the number of arbitrary coefficients involved in the general solution. In that for $g^{mn} = 1$, out of tmn coefficients following the first a proportion of $(mn - 1)/mn$ may be taken arbitrary. For $f^n = 1$, the proportion $(n - 1)/n$ are arbitrary. Having assigned f, the proportion still remaining arbitrary is

$$\frac{mn - 1}{mn} - \frac{n - 1}{n} = \frac{m - 1}{mn}.$$

Suppose $f = \varphi^{-1}\epsilon\varphi$, where $\epsilon^n = 1$, and suppose h the general solution of $h^m = \epsilon$. Then $g = \varphi^{-1}h\varphi$ is the general solution of $g^m = f$. We have then to consider the functional equation $h^m = \epsilon$.

Let $\omega^m = \epsilon$; then $\omega^{mn} = 1$. Putting mn for n and ω for ϵ in (3) we observe that $z_{km+s} = \epsilon^k z_s$ for all values of k and s. Equation (24) becomes

$$\frac{1}{m}(\omega^{-1}z_1 + \omega^{-2}z_2 + \cdots + \omega^{-m}z_n) = r.$$

For all values of $s \not\equiv 1 \pmod{n}$, we have from (25)

$$\frac{1}{m}(\omega^{-s}z_1 + \omega^{-2s}z_2 + \cdots + \omega^{-ms}z_m) = a_s r^s + a_{n+s}r^{n+s} + \cdots;$$

for all other values of s the left members of (25) vanish. Applying the method of Section 5, we have

$$z_t = \varphi_1(\omega^t r),$$

where

$$\varphi_1(r) = r + \sum_{p=0}^{\infty}\sum_{k=1}^{m-1} a_{(pm+k)n+1}r^{(pm+k)n+1}. \tag{40}$$

Hence h is defined by

$$z = \varphi_1(r), \qquad h(z) = \varphi_1(\omega r), \qquad \omega^m = \epsilon, \qquad \epsilon^n = 1.$$

Hence the most general solution for $g^m = f$ where $f^n = 1$ is given by

$$g = \varphi^{-1}h\varphi = \varphi^{-1}\varphi_1\omega\varphi_1^{-1}\varphi.\dagger$$

* This is a very special case of the more general problem, the solution of which is equivalent to finding an n-section of a curvilinear angle in the sense employed by Kasner as an extension of the idea of symmetry and corresponding bisection due to Schwarz. See G. A. Pfeiffer, "On the Conformal Geometry of Analytic Arcs," AMER. JOUR. MATH., XXXVII, 4 (1915).

† A less specific form may be obtained more directly. Suppose g in form $\lambda^{-1}\omega\lambda$. Then $\lambda^{-1}\epsilon\lambda = f = \varphi^{-1}\epsilon\varphi$, whence $\varphi\lambda^{-1}\epsilon = \epsilon\varphi\lambda^{-1}$. Putting μ for $\varphi\lambda^{-1}$ we have $\mu(\epsilon z) = \epsilon\mu(z)$; hence μ must be of the form $\mu(z) = z\theta(z^n)$. Hence if φ is given we may choose $\lambda = \mu^{-1}\varphi$ in $g = \lambda^{-1}\omega\lambda = \varphi^{-1}\mu\omega\mu^{-1}\varphi$ as the most general solution of $g^m = f$. By Theorem V it is evident that there is a sufficient number of arbitrary constants, though not indicated in the very explicit form (40).

We may similarly write down the general solution of the functional equation

$$z + f(z) + f_2(z) + \cdots + f_{n-1}(z) = 0.$$

We observe that if $1 + f + \cdots + f^{n-1} = 0$, then $f^n - 1 = 0$ and f is a periodic function. The converse is not necessarily the case, as is readily verified in the special case

$$f(z) = \epsilon z + \epsilon z^2 + \epsilon z^3 + \cdots; \qquad \epsilon^3 = 1,$$

which satisfies $f^3 = 1$ but not $1 + f + f^2 = 0$.

We note that the left member of the last of equations (25) vanishes according to this condition. The general solution is then at once written in the form

$$z = \varphi_2(r), \qquad f(z) = \varphi_2(\epsilon r),$$

where

$$\varphi_2(z) = z + \sum_{k=0}^{\infty} \sum_{s=1}^{n-2} a_{kn+s+1} z^{kn+s+1}. \tag{41}$$

Similarly, the general solutions are obtained for equations formed by equating any of the left members of (25) to zero. A similar result may be obtained when the condition (yielding $n - 1$ similar conditions) is a linear relation obtained by eliminating $s - 1$ z's after equating s of these expressions to zero.

It may be shown that, corresponding to every periodic function f, where $f_n = 1$, there are others F, for which, symbolically,

$$1 + F + F^2 + \cdots + F^{n-1} = 0.*$$

---

* Instead of the substitution (8), we may effect the substitution $z_k = \sum_{t=1}^{n} \alpha_{k,\,t} u_t$ $(k = 1, 2, \cdots, n)$, where the α's form a set of orthogonal numbers. We shall for convenience write $\alpha_{p,\,n+q} = \alpha_{p+n,\,q} = \alpha_{p,\,q}$, and choose $\alpha_{k,\,n} = 1/\sqrt{n}$ $(k = 1, 2, \cdots, n)$. The last requires $\sum_{s=1}^{n} \alpha_{s,\,r} = 0$ $(r = 1, 2, \cdots, n-1)$.

We have then n equations in n u's. Each $n - 1$ of these determine all u's in terms of u_1: $u_k = \theta_k(u_1)$, $k = 2, 3, \cdots, n$. Further, the functions so determined are unique. We now introduce the further substitution $u_k = \sum_{t=1}^{n-1} \beta_{k,\,t} v_t$, $u_n = v_n$ $(k = 1, 2, \cdots, n-1)$, where $\beta_{s,\,t} = \sum_{p=1}^{n} \alpha_{p,\,s} \alpha_{p+1,\,t}$. It is readily verified that the $(n-1)^2$ quantities β form a set of orthogonal numbers.

We now have

$$\begin{aligned}
z_k &= \sum_{t=1}^{n} \alpha_{k,\,t} u_t = \alpha_{k,\,n} v_n + \sum_{t=1}^{n-1} \left(\alpha_{k,\,t} \sum_{s=1}^{n-1} \beta_{t,\,s} v_s \right) \\
&= \alpha_{k+1,\,n} v_n + \sum_{s=1}^{n-1} \left(v_s \sum_{t=1}^{n-1} \alpha_{k,\,t} \beta_{t,\,s} \right) \\
&= \alpha_{k+1,\,n} v_n + \sum_{s=1}^{n-1} \left[v_s \sum_{p=1}^{n} \left(\alpha_{p+1,\,s} \sum_{t=1}^{n-1} \alpha_{k,\,t} \alpha_{p,\,t} \right) \right] \\
&= \sum_{s=1}^{n-1} \left[v_s \left(-\frac{1}{n} \sum_{p=1}^{n} \alpha_{p+1,\,s} + \alpha_{k+1,\,s} \right) \right] + \alpha_{k+1,\,n} v_n \\
&= \sum_{s=1}^{n} \alpha_{k+1,\,s} v_s.
\end{aligned}$$

FACTORIZATION; GROUPS OF PERIODIC TRANSFORMATIONS.

10. Kasner has considered the question whether any given transformation may be factored into two transformations each of period 2. In other

Hence z_k is the same function of the v's as z_{k+1} is of the corresponding u's. But the latter are uniquely determined by any set of $n-1$ equations. Hence the relations among the v's are the same as those among the corresponding u's. Hence the θ-functions remain unchanged under transformations of the form $u_k \to \sum_{t=1}^{n-1} \beta_{k,\,t} u_t$, $u_n \to u_n$ $(k = 1, 2, \cdots, n-1)$.

Putting $F(u_1) = \sum_{k=1}^{n-1} \beta_{1,\,k}\theta_k(u_1)$, we have $\theta_k[F(u_1)] = \sum_{p=1}^{n-1} \beta_{k,\,p}\theta_p(u_1)$. We have then

$$F_2(u_1) = \sum_{k=1}^{n-1}\left[\beta_{1,\,k}\sum_{p=1}^{n-1}\beta_{k,\,p}\theta_p(u_1)\right] = \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{k=1}^{n-1}\beta_{1,\,k}\beta_{k,\,p}\right]$$

$$= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{k=1}^{n-1}\left\{\left(\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+1,\,k}\right)\left(\sum_{t=1}^{n}\alpha_{t,\,k}\alpha_{t+1,\,p}\right)\right\}\right]$$

$$= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s,t=1}^{n}\left(\alpha_{s,\,1}\alpha_{t+1,\,p}\sum_{k=1}^{n-1}\alpha_{s+1,\,k}\alpha_{t,\,k}\right)\right]$$

$$= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\left(\alpha_{s,\,1}\alpha_{s+2,\,p}\sum_{k=1}^{n-1}\alpha^2_{s+1,\,k}\right) + \sum_{\substack{s,t=1\\ s+1\neq t}}^{n}\left(\alpha_{s,\,1}\alpha_{t+1,\,p}\sum_{k=1}^{n-1}\alpha_{s+1,\,k}\alpha_{t,\,k}\right)\right\}\right]$$

$$= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left(\frac{n-1}{n}\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p} - \frac{1}{n}\sum_{\substack{s,t=1\\ s+1\neq t}}^{n}\alpha_{s,\,1}\alpha_{t+1,\,p}\right)\right]$$

$$= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p} - \frac{1}{n}\left(\sum_{s=1}^{n}\alpha_{s,\,1}\right)\left(\sum_{t=1}^{n}\alpha_{t+1,\,p}\right)\right\}\right]$$

$$= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p}\right] = \sum_{p=1}^{n-1} a_{p,2}\,\theta_p(u_1)$$

if we introduce $a_{p,\,q} = \sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+q,\,p}$.

Similarly, we obtain

$$F_m(u_1) = \sum_{p=1}^{n-1} a_{p,\,m}\theta_p(u_1).$$

We have then

$$u_1 + \sum_{p=1}^{n-1} F_p(u_1) = u_1 + \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n-1} a_{r,s}\right] = u_1 + \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n-1}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,p}\right)\right]$$

$$= u_1 + u_1\sum_{s=1}^{n-1}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,1}\right) + \sum_{p=2}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,p}\right) - \sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t,\,p}\right\}\right]$$

$$= u_1 - u_1\sum_{t=1}^{n}\alpha^2_{t,\,1} = 0.$$

The last result also follows from the relation $F_m(u_1) = \sum_{t=1}^{n}\alpha_{t,\,1}z_{t+m}$, readily deduced. In a similar way, we may also obtain

$$\theta_k^{(m)}(u_1) = \sum_{t=1}^{n-1}\left(u_t\sum_{p=1}^{n}\alpha_{p,\,k}\alpha_{p+m,\,t}\right),$$

where

$$\theta_k^{(m)}(u_1) = \theta_k^{(m-1)}[F(u_1)], \qquad \theta'_k(u_1) = \theta_k(u_1),$$

yielding

$$\sum_{m=1}^{n}\theta_k^{(m)}(u_1) = 0,$$

from which the previous result also follows.

words, given F, can f and g be found such that, symbolically,

$$F = gf; \qquad f^2 = 1, \qquad g^2 = 1?$$

We may generalize this inquiry as follows: Given $F(z)$, can $f(z)$ and $g(z)$ be obtained where, with the notation (2),

$$F(z) = g[f(z)]; \qquad f_m(z) = z, \qquad g_n(z) = z? \tag{42}$$

We shall assume these functions to be defined as follows:

$$\begin{aligned} F(z) &= K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \\ f(z) &= \epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \qquad \epsilon_1^m = 1; \\ g(z) &= \epsilon_2 z + \mu_2 z^2 + \mu_3 z^3 + \cdots, \qquad \epsilon_2^n = 1. \end{aligned} \tag{43}$$

We shall assume here that ϵ_1 and ϵ_2 are primitive roots of unity of orders m and n, respectively. It is to be observed that all the coefficients λ and μ in (43) may be taken arbitrarily, consistent with conditions (32). For our purpose, we shall inquire more minutely as to the form of these conditions.

11. From (29′) and (27) we have the identity

$$\begin{aligned} &\epsilon_1 r + (\epsilon_1^2 a_2 r^2 + \cdots + \epsilon_1^m a_m r^m) + (\epsilon_1^2 a_{m+2} r^{m+2} + \cdots + \epsilon_1^m a_{2m} r^{2m}) + \cdots \\ &\quad = \epsilon_1[r + (a_2 r^2 + \cdots + a_m r^m) + (a_{m+2} r^{m+2} + \cdots + a_{2m} r^{2m}) + \cdots] \\ &\qquad + \lambda_2[r + (a_2 r^2 + \cdots + a_m r^m) + (a_{m+2} r^{m+2} + \cdots + a_{2m} r^{2m}) + \cdots]^2 \\ &\qquad + \lambda_3[\qquad\qquad\qquad\quad]^3 + \cdots. \end{aligned} \tag{44}$$

On equating coefficients these yield

$$\begin{aligned} &\lambda_2 = (\epsilon_1^2 - \epsilon_1)a_2, \qquad \lambda_3 = (\epsilon_1^3 - \epsilon_1)a_3 - 2(\epsilon_1^2 - \epsilon_1)a_2^2, \qquad \text{etc.;} \\ &\qquad a_2 = \frac{\lambda_2}{\epsilon_1^2 - \epsilon_1}, \qquad a_3 = \frac{\lambda_3}{\epsilon_1^3 - \epsilon_1} + \frac{2\lambda_2^2}{(\epsilon_1^3 - \epsilon_1)(\epsilon_1^2 - \epsilon_1)}, \quad \text{etc.} \end{aligned} \tag{45}$$

Similarly, we obtain corresponding relations for the coefficients μ. Equating the coefficients of z^{pm+1} in (44), and using the set (45), we readily obtain for (32)

$$\begin{aligned} \lambda_{pm+1} = {} & \frac{2\epsilon_1 - pm}{\epsilon_1^2 - \epsilon_1}\lambda_2\lambda_{pm} + \frac{3\epsilon_1^2 - (pm-1)}{\epsilon_1^3 - \epsilon_1}\lambda_3\lambda_{pm-1} \\ & + \frac{2(\epsilon_1^2 + \epsilon_1 - 1)(pm-1) - 2\epsilon_1^2(2\epsilon_1 - 1) - \frac{1}{2}(\epsilon_1 + 1)(pm-1)(pm-2)}{(\epsilon_1 + 1)(\epsilon_1^2 - \epsilon_1)^2}\lambda_2^2\lambda_{pm-1} \\ & + (\text{lower orders of } \lambda), \end{aligned} \tag{46}$$

and similar relations for μ_{sn+1}. The form (46) presupposes $m > 2$.

12. The condition $F = fg$ yields the identity

$$\begin{aligned} K_1 z + K_2 z^2 + K_3 z^3 + \cdots = \quad & \epsilon_2(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots) \\ & + \mu_2(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots)^2 \\ & + \mu_3(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots)^3 \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{47}$$

If, consistent with conditions (46), and the corresponding ones for the coefficients μ, the λ's and μ's can be chosen so as to satisfy (47) identically, then $F(z)$ can be factored into two periodic transformations of orders m and n respectively. From (47), we have as a necessary condition

$$K_1 = \epsilon_1\epsilon_2. \tag{48}$$

Hence K_1 must be a root of unity. Equating coefficients of higher powers,

$$K_2 = \epsilon_2\lambda_2 + \epsilon_1^2\mu_2, \tag{49}$$

$$K_3 = \epsilon_2\lambda_3 + 2\epsilon_1\lambda_2\mu_2 + \epsilon_1^3\mu_3, \tag{50}$$

$$K_4 = \epsilon_2\lambda_4 + (2\epsilon_1\lambda_3 + \lambda_2^2)\mu_2 + 3\epsilon_1^2\lambda_2\mu_3 + \epsilon_1^4\mu_4, \tag{51}$$

$$K_5 = \epsilon_2\lambda_5 + (2\epsilon_1\lambda_4 + 2\lambda_2\lambda_3)\mu_2 + (3\epsilon_1^2\lambda_3 + 3\epsilon_1\lambda_2^2)\mu_3 + 4\epsilon_1^3\lambda_2\mu_4 + \epsilon_1^5\mu_5, \tag{52}$$

etc. The question now is, assuming (48) to be satisfied, can the λ's and μ's always be chosen without involving necessary conditions among the K's or the indices m and n? Equating the coefficients of z^t, z^{t-1}, z^{t-2}, we obtain

$$\begin{aligned} K_t \quad = & [\epsilon_2\lambda_t + \epsilon_1^t\mu_t] + [2\epsilon_1\mu_2\lambda_{t-1} + \epsilon_1^{t-2}(t-1)\lambda_2\mu_{t-1}] \\ & + [2\lambda_2\mu_2 + 3\epsilon_1^2\mu_3]\lambda_{t-2} \\ & + \left[\epsilon_1^{t-3}(t-2)\lambda_3 + \epsilon_1^{t-4}\frac{(t-2)(t-3)}{2}\lambda_2^2\right]\mu_{t-2} \\ & + [\text{lower orders of } \lambda \text{ and } \mu], \end{aligned} \tag{53}$$

$$K_{t-1} = [\epsilon_2\lambda_{t-1} + \epsilon_1^{t-1}\mu_{t-1}] + [2\epsilon_1\mu_2\lambda_{t-2} + \epsilon_1^{t-3}(t-2)\lambda_2\mu_{t-2}] + [\text{lower orders of } \lambda \text{ and } \mu], \tag{54}$$

$$K_{t-2} = [\epsilon_2\lambda_{t-2} + \epsilon_1^{t-2}\mu_{t-2}] + [\text{lower orders of } \lambda \text{ and } \mu]. \tag{55}$$

The entire set of conditions obtained are necessary and sufficient for factorization. We observe that for each coefficient K_t in these equations *two* new quantities λ_t and μ_t are introduced, which may be taken arbitrarily except for periodic orders of these coefficients. Further, the new quantities introduced appear with non-zero coefficients. If either λ_t or μ_t is independent, the condition involving K_t can then be satisfied. Hence, the only way for a necessary condition to obtain among the coefficients K is for *both*

the coefficients λ_t and μ_t to be conditioned in terms of coefficients of lower order in the form (46). Hence t must be of the forms both $pm + 1$ and $sn + 1$. We may write

$$t = pm + 1 = sn + 1. \tag{56}$$

Hence λ_t and μ_t in (53) are not independent. These are then to be replaced by the equivalent expressions in terms of λ's and μ's of lower order, from the form (46). Hence a necessary condition for obtaining a necessary condition among the coefficients K is that it be possible to eliminate simultaneously the coefficients λ_{t-1} and μ_{t-1} from the expressions for K_t and K_{t-1} in terms of these coefficients and those of lower order. Using (56), the condition for this is

$$\frac{\epsilon_2[2\epsilon_1 - (t-1)]}{\epsilon_1^2 - \epsilon_1}\lambda_2 + 2\epsilon_1\mu_2 = \epsilon_2\left\{\frac{\epsilon_1[2\epsilon_2 - (t-1)]}{\epsilon_2^2 - \epsilon_2}\mu_2 + \epsilon_1^{-1}(t-1)\lambda_2\right\},$$

which may be written

$$(t-3)\left[\frac{\lambda_2}{\epsilon_1^2 - \epsilon_1} - \frac{\mu_2}{\epsilon_2^2 - \epsilon_2}\right] = 0. \tag{57}$$

The case $t = 3$ requires $m = n = 2$, already considered by Kasner.* We may then confine our attention to the condition

$$(\epsilon_2^2 - \epsilon_2)\lambda_2 - (\epsilon_1^2 - \epsilon_1)\mu_2 = 0, \tag{58}$$

and choose $t > 3$. If λ_2 and μ_2 can be so chosen as *not* to satisfy (58), then no necessary condition is required among the K's. The only condition upon these is given by (49). Eliminating μ_2, we have

$$K_2 = \frac{\epsilon_1\epsilon_2}{\epsilon_1^2 - \epsilon_1}(\epsilon_1\epsilon_2 - 1)\lambda_2. \tag{59}$$

Here K_2 is assigned and λ_2 completely arbitrary. Hence we may choose λ_2 so that (59) is not satisfied unless

$$K_2 = 0 \qquad \text{and} \qquad \epsilon_1\epsilon_2 - 1 = 0. \tag{60}$$

The latter condition requires $K_1 = 1$. Hence we have the result

If $K_1 \neq 1$, the transformation $F(z) = K_1 z + \cdots$ can always be factored into two transformations of order m and n, the orders being so chosen that $K_1 = \epsilon_1\epsilon_2$, where ϵ_1 and ϵ_2 are primitive roots of unity of order m and n respectively.

13. Suppose now that $K_1 = \epsilon_1\epsilon_2 = 1$. Then we must have

$$m = n, \tag{61}$$

* In this case, $\epsilon_1 = \epsilon_2 = -1$, (45) yield $\lambda_3 = -\lambda_2^2$, $\mu_3 = -\mu_2$, and (49) and (50) give $K_3 - K_2 = 0$ as a necessary condition, (48) requiring $K_1 = 1$. Cf. Kasner, loc. cit.

since ϵ_1 and ϵ_2 are primitive roots. Taking further $K_2 = 0$, we have

$$\mu_2 = -\epsilon_2^3\lambda_2. \tag{62}$$

If we now write (46) and the corresponding expression in μ's in the form

$$\lambda_{pm+1} = \frac{2\epsilon_1 - pm}{\epsilon_1^2 - \epsilon_1}\lambda_2\lambda_{pm} + A\lambda_3\lambda_{pm-1} + B\lambda_2^2\lambda_{pm-1} + [\text{lower orders of } \lambda], \tag{63}$$

$$\mu_{sn+1} = \frac{2\epsilon_2 - sn}{\epsilon_2^2 - \epsilon_2}\mu_2\mu_{sn} + C\mu_3\mu_{sn-1} + D\mu_2^2\mu_{sn-1} + [\text{lower orders of } \mu],$$

and eliminate λ_{t-1} and μ_{t-1} from equations (53) and (54), we have, on using (56), (60), (61) and (62),

$$\begin{aligned} K_t + \frac{t-3}{\epsilon_1^2 - \epsilon_1}\lambda_2 K_{t-1} &= \left[\epsilon_2 A\lambda_3 + \epsilon_2 B\lambda_2^2 - 2\epsilon_2^3\lambda_2^2 + 3\epsilon_1^2\mu_3 - \frac{2\epsilon_2^2(t-3)}{\epsilon_1^2 - \epsilon_1}\lambda_2^2\right]\lambda_{t-2} \\ &\quad + \left[\epsilon_1 C\mu_3 + \epsilon_2^5 D\lambda_2^2 + \epsilon_2^2(t-2)\lambda_3 + \epsilon_2^3\frac{(t-2)(t-3)}{2}\lambda_2^2 + \frac{\epsilon_1^2(t-2)(t-3)}{\epsilon_1^2 - \epsilon_1}\right]\mu_{t-2} \\ &\quad + [\text{lower orders of } \lambda \text{ and } \mu]. \end{aligned} \tag{64}$$

Equation (55) becomes

$$K_{t-2} = [\epsilon_2\lambda_{t-2} + \epsilon_2\mu_{t-2}] + [\text{lower orders of } \lambda \text{ and } \mu]. \tag{65}$$

The form of (64) presupposes $t > 5$. For the moment we shall assume this satisfied. Here λ_{t-2} and μ_{t-2} in (64) and (65) can be so chosen as to satisfy both, unless the determinant of these coefficients vanishes. The condition for this becomes

$$[\epsilon_2 A - \epsilon_2^2(t-2)]\lambda_3 + [3\epsilon_1^2 - \epsilon_1 C]\mu_3 + \frac{2\epsilon_2(t-5)}{\epsilon_1^2 - 1}\lambda_2^2 = 0.$$

Here λ_2 is independent and the only condition upon λ_3 and μ_3 is (50). Eliminating μ_3 we obtain

$$L\lambda_3 + M\lambda_2^2 + NK_3 = 0, \tag{66}$$

where L, M and N are rational expressions in t and ϵ_2. We have

$$N = \epsilon_2\frac{t-5}{\epsilon_2^2 - 1} \neq 0.$$

In (66), λ_2 and λ_3 are completely independent. Hence these may be chosen

so as not to satisfy (66), unless

$$L = M = 0 \tag{67}$$

identically, and $K_3 = 0$. On substituting their values for A and C, we find these conditions fulfilled. Hence if $K_1 = 1$, and $K_2 = K_3 = 0$, we must resort to further analysis.

14. For this purpose we return to the identity (44). If t be *even*, we may write the general condition in the form

$$\begin{aligned} \epsilon_1^t a_t = {} & \epsilon_1 a_t + \lambda_t \\ & + \lambda_2 M_{t-1} + \lambda_{t-1} M_2 \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{k+1} M_{t-k} + \lambda_{t-k} M_{k+1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{t/2} M_{(t/2)+1} + \lambda_{(t/2)+1} M_{t/2}, \end{aligned} \tag{68}$$

where

$$M_{t-k} = (k+1)a_{t-k} + \text{(polynomial in lower orders of } a). \tag{69}$$

If t be *odd*, we have the form

$$\begin{aligned} \epsilon_1^t a_t = {} & \epsilon_1 a_t + \lambda_t \\ & + \lambda_2 M_{t-1} + \lambda_{t-1} M_2 \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{k+1} M_{t-k} + \lambda_{t-k} M_{k+1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{(t-1)/2} M_{(t+3)/2} + \lambda_{(t+3)/2} M_{(t-1)/2} \\ & + \lambda_{(t+1)/2} M_{(t+1)/2}. \end{aligned} \tag{70}$$

From (45), we have a_2 and a_3 expressed in terms of the corresponding λ's. Substituting in (69) for $t = 4$, we have a_4 in terms of λ's. Using (68) and (70), we finally have all a's that appear in (44) expressed in the forms (68) and (70) where the M's are expressed in λ's in the form

$$\begin{aligned} M_{t-k} &= \frac{k+1}{\epsilon_1^{t-k} - \epsilon_1} \lambda_{t-k} + \text{(polynomial in lower orders of } \lambda), \\ M_{k+1} &= \frac{t-k}{\epsilon_1^{k+1} - \epsilon_1} \lambda_{k+1} + \text{(polynomial in lower orders of } \lambda). \end{aligned} \tag{71}$$

If $t - k \equiv 1 \pmod{m}$, the first term indicated in (71) is of course absent. The highest order of λ that multiplies λ_{t-k} is λ_{k+1}, and occurs only in the products $\lambda_{k+1} M_{t-k}$ and $\lambda_{t-k} M_{k+1}$. If $t - k > (t+1)/2$, then λ_{t-k} occurs only to the first power in (68) or (70). The coefficient of λ_{t-k} is

$$\left(\frac{k+1}{\epsilon_1^{t-k} - \epsilon_1} + \frac{t-k}{\epsilon_1^{k+1} - \epsilon_1} \right) \lambda_{k+1} + \text{(polynomial in lower orders of } \lambda).$$

If now we take $t-1$ a multiple of m, we obtain from (68), if t be even,

$$\lambda_t = N_2\lambda_{t-1} + N_3\lambda_{t-2} + \cdots + N_{k+1}\lambda_{t-k} + \cdots + N_{t/2}\lambda_{(t/2)+1} + (\text{polynomial in lower orders of } \lambda), \tag{72}$$

and from (70), if t be odd,

$$\lambda_t = N_2\lambda_{t-1} + N_3\lambda_{t-2} + \cdots + N_{k+1}\lambda_{t-k} + \cdots + \tfrac{1}{2}N_{(t+1)/2}\lambda_{(t+1)/2} + (\text{polynomial in lower orders of } \lambda), \tag{73}$$

where

$$N_{k+1} = \frac{\epsilon_1^k(k+1) - (t-k)}{\epsilon_1^{k+1} - \epsilon_1}\lambda_{k+1} + (\text{polynomial in lower orders of } \lambda). \tag{74}$$

If $k \equiv 0 \pmod{m}$, then the term of order $k+1$ in (74) is missing. If $t = m+1$, every term is present. The relations (72) and (73) are simply more explicit forms of the necessary conditions (32). In similar fashion, we may now write out the corresponding expressions in the μ's, thus obtaining the necessary relations among the coefficients of f and g in (43).

15. We shall now write (53) in a form corresponding to (72) and (73), and then apply once more the method of Sections 12 and 13. Referring to the identity (47) we may write, if t is *even*,

$$\begin{aligned} K_t = {} & \epsilon_2\lambda_t + \epsilon_1^t\mu_t \\ & + P_{t,\,2}\lambda_{t-1} + Q_{t,\,2}\mu_{t-1} \\ & + \cdots \\ & + P_{t,\,k+1}\lambda_{t-k} + Q_{t,\,k+1}\mu_{t-k} \\ & + \cdots \\ & + P_{t,\,t/2}\lambda_{(t/2)+1} + Q_{t,\,t/2}\mu_{(t/2)+1} \\ & + (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t/2)+1), \end{aligned} \tag{75}$$

and if t is *odd*,

$$\begin{aligned} K_t = {} & \epsilon_2\lambda_t + \epsilon_1^t\mu_t \\ & + P_{t,\,2}\lambda_{t-1} + Q_{t,\,2}\mu_{t-1} \\ & + \cdots \\ & + P_{t,\,k+1}\lambda_{t-k} + Q_{t,\,k+1}\mu_{t-k} \\ & + \cdots \\ & + P_{t,\,(t-1)/2}\lambda_{(t+3)/2} + Q_{t,\,(t-1)/2}\mu_{(t+3)/2} + Q_{t,\,(t+1)/2}\mu_{(t+1)/2} \\ & + (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2), \end{aligned} \tag{76}$$

where

$$P_{t,\,k+1} = (k+1)\epsilon_1^k\mu_{k+1} + (\text{lower orders of } \lambda \text{ and } \mu) \tag{77}$$

and

$$Q_{t,\,k+1} = (t-k)\epsilon_1^{t-k-1}\lambda_{k+1} + (\text{lower orders of } \lambda). \tag{78}$$

All λ's and μ's in (75) and (76) are independent except those of order $pm+1$ or $sn+1$. We now examine the cases determined by (56), (60),

(61), and (62). In that event λ_t and μ_t in (75) and (76) must be replaced by the equivalent expressions of the form (72) or (73). If $t = n + 1$, then all the remaining λ's and μ's in (75) or (76) are independent. The coefficient of λ_{k+1} in (78) becomes $(t - k)\epsilon_1^{-k}$, or $(t - k)\epsilon_2^k$.

Substituting for λ_t and μ_t in (75) their equivalent expressions, we have for t even

$$\begin{aligned} K_t = {} & E_2\lambda_{t-1} + F_2\mu_{t-1} \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + E_{k+1}\lambda_{t-k} + F_{k+1}\mu_{t-k} \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + E_{t/2}\lambda_{(t/2)+1} + F_{t/2}\mu_{(t/2)+1} \\ & + (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t/2) + 1), \end{aligned} \tag{79}$$

and from (76) for t odd

$$\begin{aligned} K_t = {} & E_2\lambda_{t-1} + F_2\mu_{t-1} \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + E_{k+1}\lambda_{t-k} + F_{k+1}\mu_{t-k} \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + E_{(t-1)/2}\lambda_{(t+3)/2} + F_{(t-1)/2}\mu_{(t+3)/2} \\ & + \left(\frac{\epsilon_2}{2}N_{(t+1)/2}\lambda_{(t+1)/2} + Q_{t,\,(t+1)/2}\mu_{(t+1)/2} + \frac{\epsilon_1}{2}T_{(t+1)/2}\mu_{(t+1)/2}\right) \\ & + (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t + 1)/2), \end{aligned} \tag{80}$$

where

$$E_{k+1} = P_{t,\,k+1} + \epsilon_2 N_{k+1}; \qquad F_{k+1} = Q_{t,\,k+1} + \epsilon_1 T_{k+1}, \tag{81}$$

where T is the expression in μ's corresponding to N in (74), and has the form

$$T_{k+1} = \frac{\epsilon_2^k(k + 1) - (t - k)}{\epsilon_2^{k+1} - \epsilon_2}\mu_{k+1} + (\text{polynomial in lower orders of } \mu). \tag{82}$$

16. From the form of (75) and (76) we may write

$$\begin{aligned} K_{t-1} = {} & \epsilon_2\lambda_{t-1} + \mu_{t-1} \\ & + P_{t-1,\,2}\lambda_{t-2} + Q_{t-1,\,2}\mu_{t-2} \\ & + P_{t-1,\,3}\lambda_{t-3} + Q_{t-1,\,3}\mu_{t-3} \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot, \end{aligned} \tag{83}$$

and similar expressions for the lower orders of K.

Comparing (83) with (79) or (80), we observe that K_t may be taken independent of the lower orders of K unless

$$\begin{vmatrix} E_2 & F_2 \\ \epsilon_2 & 1 \end{vmatrix} = 0 \tag{84}$$

identically. As already seen, this reduces to (57), involving (60), (61) and (62). Assuming (84) satisfied, and eliminating λ_{t-1} and μ_{t-1} from the expressions for K_t and K_{t-1}, the eliminant and the expression for K_{t-2} can always be satisfied by a proper choice of the λ's and μ's involved unless λ_{t-2} and μ_{t-2} can be eliminated simultaneously. This obviously requires that the coefficients of λ_{t-1}, μ_{t-1}, λ_{t-2}, and μ_{t-2} in the expressions for K_t, K_{t-1} and K_{t-2} shall be linearly dependent. The condition for this is that all 3-rowed determinants of the matrix

$$\left\| \begin{matrix} E_2 & F_2 & E_3 & F_3 \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} \end{matrix} \right\| \tag{85}$$

vanish.

Assuming (84) satisfied, the second column of (85) may be replaced by 0's, leaving but one determinant for consideration. Since E_3 and F_3 are the only expressions containing λ's and μ's of order 3, we have as a necessary condition, on expanding according to the minors of the elements of the first column and using (62),

$$\epsilon_1^{-1}E_3 - \epsilon_2 F_3 + (\text{polynomial in } \lambda_2) = 0,$$

the λ-polynomial being in fact λ_2^2 multiplied by a constant. Using (74), (82), (77), (78) and (81), this reduces to

$$\frac{t-5}{\epsilon_1^2 - 1}(\epsilon_2\lambda_3 + \epsilon_1^3\mu_3) + (\text{polynomial in } \lambda_2) = 0. \tag{86}$$

Here λ_2 is completely independent and λ_3 and μ_3 restricted only by the relation (50). Assuming for the moment that $t \neq 5$, and using (50), the last equation becomes

$$K_3 + (\text{polynomial in } \lambda_2) = 0. \tag{87}$$

Here K_3 is assigned. Hence λ_2 may be so chosen that (88) is *not* satisfied, unless $K_3 = 0$ and the polynomial in λ_2 in (87) vanishes identically. Both of these conditions are necessary if there is to be a necessary relation among the K's. In Section 13 we found that the polynomial of (87) does vanish identically.

17. Assuming now $K_2 = K_3 = 0$, and following the same reasoning, λ_{t-1}, λ_{t-2}, λ_{t-3} and the corresponding μ's may be chosen so that the conditions for K_t, K_{t-1}, K_{t-2} and K_{t-3} are all satisfied unless the coefficients of the λ's and μ's in these expressions are linearly dependent. As before, the condition for this is that the matrix

$$\left\| \begin{matrix} E_2 & F_2 & E_3 & F_3 & E_4 & F_4 \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} & P_{t-1,\,3} & Q_{t-1,\,3} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} & P_{t-2,\,2} & Q_{t-2,\,2} \\ 0 & 0 & 0 & 0 & \epsilon_2 & \epsilon_1^{-2} \end{matrix} \right\| \tag{88}$$

be of rank < 4. Since in (84) and (85) the last columns are linearly expressible in terms of the previous columns, we may replace columns 2 and 4 in (88) by 0's, leaving but one 4-rowed determinant for consideration. In (88), E_4 and F_4 are the only elements involving λ's and μ's of order 4. Expanding in terms of the determinants of the last two columns, and replacing μ_2 and μ_3 by their equivalent expressions in λ by putting $K_2 = K_3 = 0$ in (49) and (50),

$$\epsilon_2^2 \begin{vmatrix} E_4 & F_4 \\ \epsilon_2 & \epsilon_1^{-2} \end{vmatrix} + (\text{polynomial in } \lambda\text{'s of order } < 4) = 0.$$

This reduces to

$$\frac{t-7}{\epsilon_1^3 - 1}(\epsilon_2\lambda_4 + \epsilon_1^4\mu_4) + (\text{polynomial in } \lambda\text{'s of order } < 4) = 0. \tag{89}$$

If $t \neq 7$, this becomes, on using (51),

$$K_4 + (\text{polynomial in } \lambda\text{'s of order } < 4) = 0. \tag{90}$$

As before, the λ's can be chosen so that (90) is not satisfied, unless $K_4 = 0$ and the λ-polynomial vanishes identically. If $K_4 = 0$, (51) gives μ_4 also in terms of λ's.

Suppose now that the condition that K_t depend for its value on K's of lower order requires

$$K_2 = K_3 = K_4 = \cdots = K_k = 0. \tag{91}$$

These determine $\mu_2, \mu_3, \cdots, \mu_k$ as polynomials in λ's of corresponding and lower orders. Continuing as before, we find as another necessary condition that the matrix

$$\begin{Vmatrix} E_2 & F_2 & E_3 & F_3 & \cdots & E_{k+1} & F_{k+1} \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} & \cdots & P_{t-1,\,k} & Q_{t-1,\,k} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} & \cdots & P_{t-2,\,k-1} & Q_{t-2,\,k-1} \\ 0 & 0 & 0 & 0 & \cdots & P_{t-3,\,k-2} & Q_{t-3,\,k-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \epsilon_2 & \epsilon_1^{1-k} \end{Vmatrix} \tag{92}$$

be of rank $< k + 1$. As before, every second column except the last may be replaced by 0's, leaving but a single $(k+1)$-rowed determinant for consideration. Expanding according to the elements of the last two columns we obtain

$$\epsilon_2^{k-1} \begin{vmatrix} E_{k+1} & F_{k+1} \\ \epsilon_2 & \epsilon_1 \end{vmatrix} + (\text{polynomial in } \lambda\text{'s of order } < k+1) = 0,$$

which reduces to

$$\frac{t-2k-1}{\epsilon_1^k - 1}(\epsilon_2\lambda_{k+1} + \epsilon_1^{k+1}\mu_{k+1}) + (\text{polynomial in } \lambda\text{'s of order } < k+1) = 0,$$

or

$$\frac{t-2k-1}{\epsilon_1^k-1}K_{k+1}+\text{(polynomial in }\lambda\text{'s of order }<k+1)=0. \quad (93)$$

18. We shall now first restrict ourselves to the case $t=n+1$. Hence all the terms indicated in (74) and (82) are present, and all λ's and μ's in (75) or (76), except λ_t and μ_t, are independent. We shall find it convenient to consider two cases, according as t is *even* or *odd*.

We shall first consider t *even*. Then $t\neq 2k+1$ in any case. From the form of (79), we may continue the method of the previous sections up to the case $k=t/2$. Hence, if the condition $K_2=K_3=\cdots=K_k=0$ has been shown necessary, we obtain from (93) the condition $K_{k+1}=0$ and the further condition that the polynomial in λ vanish identically. Hence we have as a necessary condition that K_t be conditioned in terms of lower orders of K

$$K_2=K_3=\cdots=K_{t/2}=0. \quad (94)$$

If the polynomials indicated in (87) and (90) and that resulting from (93) do not vanish, then of course all K's are independent. However, we shall presently show by an indirect method that they all vanish identically. From (94) and the relations of Section 12, we have all μ's up to order $t/2$ determined as polynomials in λ's of corresponding and lower orders.

Conditions (94) and the vanishing of the polynomials considered above suffice for the elimination of the λ's and μ's of orders $(t/2)+1$ to $t-1$ from the expressions for K_t, K_{t-1}, $\cdots$, $K_{(t/2)+1}$. From the method of procedure in Section 13 we observe that, if we eliminate λ_{t-1} and μ_{t-1} from between K_t and K_{t-1} and then λ_{t-2} and μ_{t-2} from between the result and K_{t-2}, etc., the result of the entire elimination of λ's and μ's of orders $t-1$ down to order $(t/2)+1$ takes the form

$$K_t+H_2K_{t-1}+H_3K_{t-2}+\cdots+H_{t/2}K_{(t/2)+1}=P(\lambda), \quad (95)$$

where $P(\lambda)$ is a polynomial in λ's of order $<(t/2)+1$. It is also clear that no λ of order $>t/2$ appears in the left member of (95). From the form of (79), (83), etc., we observe that the highest orders of λ and μ in the coefficients E and F following λ_{t-1} and μ_{t-1} are not affected by the elimination of these variables; that those following λ_{t-2} and μ_{t-2} are not affected by the next elimination, etc. In other words, the term of highest order in H_{k+1} in (95) is ϵ_1 times the corresponding term in the coefficient of λ_{t-k} in (79). In other words,

$$H_{k+1}=\frac{t-2k-1}{\epsilon_1^{k+1}-\epsilon_1}\lambda_{k+1}+\text{(polynomial in }\lambda\text{'s of lower order)}. \quad (96)$$

The coefficient of λ_{k+1} in (96) is, furthermore, not zero. In the left member of (95), $\lambda_{t/2}$ occurs only in $H_{t/2}$. If it occurs in $P(\lambda)$, it is multiplied by some variable λ. Hence if $K_{(t/2)+1} \neq 0$, the λ's of order $< t/2$ may be assigned arbitrarily and $\lambda_{t/2}$ assigned a value to satisfy (95) for *all* values of the K's. Hence for a *necessary* relation to obtain among the K's, we must have $K_{(t/2)+1} = 0$. Similarly, we must have $K_{(t/2)+2} = 0$, etc. Hence the only way in which a necessary condition must subsist among the K's is to have

$$K_{(t/2)+1} = K_{(t/2)+2} = \cdots = K_{t-1} = 0.$$

If, now, $P(\lambda)$ in (95) vanish identically, the necessary resulting condition would be

$$K_t = 0. \tag{97}$$

Hence, if t is even and $= n + 1$, the only condition that may be required among the K's is to have $K_t = 0$, and this can result only if

$$K_2 = K_3 = \cdots = K_{t-1} = 0,$$

and $P(\lambda)$ vanish identically.

These are, furthermore, sufficient conditions if the polynomials resulting from (93) also vanish identically. If the procedure be followed out in detail for $t = 4$, we obtain (95) in the form

$$K_4 + \frac{\epsilon_1^2 \lambda_2}{\epsilon_1 - 1} K_3 = 0. \tag{98}$$

For the case $t = 6$ we obtain, without any essential difficulty,

$$K_6 + \frac{3\lambda_2}{\epsilon_1^2 - \epsilon_1} K_5 + \left[\frac{\epsilon_1^4}{\epsilon_1^2 - 1} \lambda_3 + \frac{2\epsilon_1^3(\epsilon_1 + 2)}{(\epsilon_1 - 1)(\epsilon_1^2 - 1)} \lambda_2^2 \right] K_4 = 0. \tag{99}$$

19. Suppose now that t is *odd*. The method of Section 17 can be applied so long as t remains $> 2k + 1$. Hence we obtain as necessary conditions for dependence among the K's

$$K_2 = K_3 = \cdots = K_{(t-1)/2} = 0.$$

Also, from the preceding discussion, the elimination of K's of all orders down to $K_{(t+3)/2}$ does not affect the leading terms of order $(t + 1)/2$ in (80). Evaluating the corresponding expressions in (80),

$$\begin{aligned} \frac{\epsilon_2}{2} N_{(t+1)/2} \lambda_{(t+1)/2} + Q_{t,\,(t+1)/2} \mu_{(t+1)/2} + \frac{\epsilon_1}{2} T_{(t+1)/2} \mu_{(t+1)/2} \\ = \frac{t+1}{4} (\epsilon_2 \lambda_{(t+1)/2} - \epsilon_1 \mu_{(t+1)/2})^2 + \lambda_{(t+1)/2} \text{ (lower orders)} \\ + \mu_{(t+1)/2} \text{ (lower orders)}. \end{aligned} \tag{100}$$

From the form of (75) and (76),

$$K_{(t+1)/2} = \epsilon_2\lambda_{(t+1)/2} - \epsilon_1\mu_{(t+1)/2} + (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2). \tag{101}$$

Hence the result of the elimination of K's down to order $(t+3)/2$ takes the form

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2} = \lambda_{(t+1)/2}\,(\text{lower orders}) + \mu_{(t+1)/2}\,(\text{lower orders}) + (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2). \tag{102}$$

Here $\lambda_{(t+1)/2}$ and $\mu_{(t+1)/2}$ are not independent, being connected by (101). The question now arises, can these variables be eliminated from between (102) and (101)? For this purpose we have to evaluate the coefficients of $\lambda_{(t+1)/2}$ and $\mu_{(t+1)/2}$ in (102).

20. We shall, for convenience, introduce

$$e = \frac{t+1}{2}, \tag{103}$$

whence

$$t = 2e - 1; \qquad \epsilon_1^{e-1} = \epsilon_2^{e-1} = -1. \tag{104}$$

We shall now trace back all products of the form $\lambda_e\lambda_{e-1}$, $\lambda_e\mu_{e-1}$, $\mu_e\lambda_{e-1}$, $\mu_e\mu_{e-1}$, occurring in the process of obtaining (102). In the following, all expressions involving such products will be included.

From (44), we have the general form

$$\begin{aligned}\epsilon^t a_t = {} & \epsilon a_t + \lambda_t \\ & + \lambda_2[2a_{t-1} + \cdots + 2a_{e-1}a_{e+1}] \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + \lambda_{e-1}[(e-1)a_{e+1} + (e-1)(e-2)a_2a_e + (\text{lower orders})] \\ & + \lambda_e[ea_e + e(e-1)a_2a_{e-1} + (\text{lower orders})] \\ & + \lambda_{e+1}[(e+1)a_{e-1} + (\text{lower orders})] \\ & + \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ & + \lambda_{t-1}[(t-1)a_2].\end{aligned} \tag{105}$$

From the general form (105) we also have

$$(\epsilon^{t-1} - \epsilon)a_{t-1} = \lambda_{t-1} + \cdots + \lambda_{e-1}[(e-1)a_e + (\text{lower orders})] + \lambda_e[ea_{e-1} + (\text{lower orders})] + \cdots; \tag{106}$$

$$(\epsilon^{e+1} - \epsilon)a_{e+1} = \lambda_{e+1} + \lambda_2[2a_e + (\text{lower orders})] + \cdots + \lambda_e ea_2; \tag{107}$$

$$(\epsilon^{e} - \epsilon)a_{e} = \lambda_e + \lambda_2[2a_{e-1} + (\text{lower orders})] + \cdots + \lambda_{e-1}(e-1)a_2; \tag{108}$$

$$(\epsilon^{e-1} - \epsilon)a_{e-1} = \lambda_{e-1} + (\text{lower orders}). \tag{109}$$

Formulas (105) and (107) presuppose $t > 5$. The case $t = 5$ will be considered specially.

Using (106), (107), (108), (109) and (45), we have from (105)

$$\begin{aligned}\lambda_t = {} & \lambda_2\lambda_{t-1}\left[\frac{2}{\epsilon_1 - 1} - \frac{t-1}{\epsilon_1(\epsilon_1 - 1)}\right] \\ & + \cdots + \lambda_{e-1}\lambda_{e+1}\left[\frac{e}{\epsilon_1} + \frac{\epsilon_1 - 1}{\epsilon_1(\epsilon_1 + 1)}\right] \\ & + \frac{e}{2\epsilon_1}\lambda_e^2 + \lambda_2\lambda_e\lambda_{e-1}\left[\frac{2e(e-1)}{\epsilon_1^2(\epsilon_1 - 1)} - \frac{2e(2\epsilon_1 + 1) - 4}{\epsilon_1(\epsilon_1^2 - 1)}\right] + \cdots.\end{aligned} \tag{110}$$

We may now proceed to the corresponding expression for μ_t. Since K_{e-1} is presumed zero,

$$0 = \epsilon_2\lambda_{e-1} - \mu_{e-1} + \text{(lower orders)},$$

whence

$$\mu_{e-1} = \epsilon_2\lambda_{e-1} + \text{(lower orders of } \lambda). \tag{111}$$

We may also write

$$K_{e+1} = [\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}] - \lambda_2[2\epsilon_2^2\lambda_e + e\mu_e] + \text{(lower orders } \lambda), \tag{112}$$

$$K_e = [\epsilon_2\lambda_e - \epsilon_1\mu_e] - (e+1)\epsilon_2^2\lambda_2\lambda_{e-1} + \text{(lower orders } \lambda), \tag{113}$$

$$\begin{aligned}K_e^2 = {} & [\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 - 2\epsilon_2^2(e+1)\lambda_2\lambda_{e-1}[\epsilon_2\lambda_e - \epsilon_1\mu_e] \\ & + \text{(lower orders } \lambda).\end{aligned} \tag{114}$$

The expression for μ_t becomes, since $\epsilon_1\epsilon_2 = 1$,

$$\begin{aligned}\mu_t = {} & \lambda_2\mu_{t-1}\left[\frac{2\epsilon_2^2}{\epsilon_1 - 1} - \frac{\epsilon_2(t-1)}{\epsilon_1 - 1}\right] + \cdots + \lambda_{e-1}\lambda_{e+1}\left[e - \frac{\epsilon_1 - 1}{\epsilon_1 + 1}\right] \\ & + \frac{\epsilon_1 e}{2}\mu_e^2 + \epsilon_2\lambda_2\lambda_{e-1}\mu_e\left[\frac{2e(e-1)}{\epsilon_1 - 1} - \frac{2e(2\epsilon_2 + 1) - 4}{\epsilon_1^2 - 1}\right] + \cdots.\end{aligned} \tag{115}$$

21. Referring now to (47), we may write

$$\begin{aligned}K_t = {} & [\epsilon_2\lambda_t + \epsilon_1\mu_t] + [-2\epsilon_2^2\lambda_2\lambda_{t-1} + \epsilon_2(t-1)\lambda_2\mu_{t-1}] + \cdots \\ & - \epsilon_2^3\lambda_2[2\lambda_{e-1}\lambda_e + \cdots] + \cdots \\ & + \mu_{e+1}[(e+1)\epsilon_1^e\lambda_{e-1} + \cdots] \\ & + \mu_e[e\epsilon_1^{e-1}\lambda_e + e(e-1)\epsilon_1^{e-2}\lambda_2\lambda_{e-1} + \cdots] \\ & + \mu_{e-1}[(e-1)\epsilon_1^{e-2}\lambda_{e+1} + (e-1)(e-2)\epsilon_1^{e-3}\lambda_2\lambda_e + \cdots] + \cdots,\end{aligned}$$

which may be written

$$\begin{aligned}K_t = {} & [\epsilon_2\lambda_t + \epsilon_1\mu_t] + \lambda_2[-2\epsilon_2^2\lambda_{t-1} + \epsilon_2(t-1)\mu_{t-1}] + \cdots \\ & + \lambda_{e-1}[-\epsilon_2^2(e-1)\lambda_{e+1} - \epsilon_1(e+1)\mu_{e+1}] - e\lambda_e\mu_e \\ & + \lambda_2\lambda_{e-1}\{\lambda_e[-2\epsilon_2^3 - \epsilon_2^3(e-1)(e-2)] \\ & + \mu_e[-\epsilon_2 e(e-1)]\} + \cdots.\end{aligned} \tag{116}$$

The only other K involving like products is

$$K_{t-1} = [\epsilon_2\lambda_{t-1} + \mu_{t-1}] + \cdots + \lambda_{e-1}[-\epsilon_2^2(e-1)\lambda_e - e\lambda_e] + \cdots. \tag{117}$$

Substituting now from (110) and (115) in (116),

$$K_t = -\epsilon_2 \frac{t-3}{\epsilon_1 - 1}\lambda_2[\epsilon_2\lambda_{t-1} + \mu_{t-1}] + \cdots$$
$$+ \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + \frac{2\lambda_{e-1}}{\epsilon_1 + 1}[\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}]$$
$$+ \epsilon_2^2\lambda_2\lambda_{e-1}\left\{\begin{array}{l} \epsilon_2\lambda_e\left[-\dfrac{\epsilon_1 - 3}{\epsilon_1 - 1}e^2 - \dfrac{\epsilon_1^2 + 4\epsilon_1 + 5}{\epsilon_1^2 - 1}e \right. \\ \qquad \left. + \dfrac{4}{\epsilon_1^2 - 1}(-\epsilon_1^2 + \epsilon_1 + 1)\right] \\ + \epsilon_1\mu_e\left[\dfrac{\epsilon_1 + 1}{\epsilon_1 - 1}e^2 - \dfrac{\epsilon_1^2 + 4\epsilon_1 + 5}{\epsilon_1^2 - 1}e \right. \\ \qquad \left. + \dfrac{4\epsilon_1}{\epsilon_1^2 - 1}\right] \end{array}\right\} + \cdots. \tag{118}$$

From (118) and (117), we now have

$$K_t + \frac{\epsilon_2(t-3)}{\epsilon_1 - 1}\lambda_2 K_{t-1} = [\text{orders} < t-1 \text{ and} > e+1]$$
$$+ \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + \frac{2\lambda_{e-1}}{\epsilon_1 + 1}[\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}]$$
$$+ \epsilon_2^2\lambda_2\lambda_{e-1}\left\{\begin{array}{l} \epsilon_2\lambda_e\left[-e^2 - \dfrac{\epsilon_1^2 - 2\epsilon_1 - 1}{\epsilon_1^2 - 1}e \right. \\ \qquad \left. - \dfrac{4\epsilon_1^2}{\epsilon_1^2 - 1}\right] \\ + \epsilon_1\mu_e\left[e^2 - \dfrac{\epsilon_1^2 + 1}{\epsilon_1^2 - 1}e + \dfrac{4\epsilon_1}{\epsilon_1^2 - 1}\right] \end{array}\right\} + \cdots. \tag{119}$$

We may observe that the right member of (119) is not affected by the elimination of any more λ's and μ's down to those of order $e+1$. From (96),

$$H_{(t-1)/2} = -\frac{2\lambda_{e-1}}{\epsilon_1 + 1} + (\text{lower orders of } \lambda). \tag{120}$$

In the process of obtaining (102) we have then, from the form (119) and (112),

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2}$$
$$= \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + [\text{polynomial in } \lambda\text{'s of order} < e] \tag{121}$$
$$- \frac{\epsilon_2^2\lambda_2\lambda_{e-1}}{\epsilon_1^2 - 1}[e^2(\epsilon_1^2 - 1) + (\epsilon_1^2 - 2\epsilon_1 - 1)e + 4\epsilon_1][\epsilon_2\lambda_e - \epsilon_1\mu_e].$$

From (121) and (114),

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2} = \epsilon_2\frac{t-3}{\epsilon_1^2-1}\lambda_2\lambda_{e-1}[\epsilon_2\lambda_e - \epsilon_1\mu_e] + (\text{polynomial in } \lambda\text{'s of order} < e). \tag{122}$$

Using (113), (122) may now be written

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2} - \epsilon_2\frac{t-3}{\epsilon_1^2-1}\lambda_2\lambda_{(t-1)/2}K_{(t+1)/2} = P(\lambda), \tag{123}$$

where $P(\lambda)$ is some polynomial in λ's of order $< (t+1)/2$. Equation (123) corresponds to (102). For the special case $t = 5$, we obtain readily

$$K_5 + (\epsilon_1 - 1)\lambda_2K_4 - \frac{3}{2}K_3^2 - \frac{\epsilon_1}{2}\lambda_2^2K_3 = 0. \tag{124}$$

In (123), $\lambda_{(t-1)/2}$ occurs in the left member only in the coefficients of $K_{(t+3)/2}$ and $K_{(t+1)/2}$. In the latter it occurs in product with the independent λ_2. Hence, reasoning as before, (123) can be made to hold for all K's, unless $K_{(t+1)/2} = 0$ and $K_{(t+3)/2} = 0$. Similarly, we must require

$$K_{(t+5)/2} = \cdots = K_{t-1} = 0.$$

Only in that event, and if $P(\lambda) = 0$ identically, can we obtain the necessary condition $K_t = 0$.

We have seen that if $K_1 \neq 1$ in (43), then F is always factorable as the product of two periodic transformations. If $K_1 = 1$ and F is factorable, the periods of f and g in (43) must be equal, giving $m = n$. We may now state the result

If the transformation

$$F(z) = z + K_2z^2 + K_3z^3 + \cdots$$

is to be factored into two transformations of period n, then the only condition that may be found necessary among the K's is of the form

$$K_{sn+1} = \textit{linear expression in K's of lower order;}$$

if s is to $= 1$, *the only way in which the condition can arise is to have*

$$K_2 = K_3 = \cdots = K_n = 0,$$

in which event the condition is

$$K_{n+1} = 0.$$

In other words, if $F(z)$ takes the form

$$z + K_r z^r + \cdots,$$

then F can always be factored into two transformations of period $> r - 1$.

It should be remembered that we have not as yet shown that if the condition

$$K_2 = K_3 = \cdots = K_n = 0 \tag{125}$$

holds, then $K_{n+1} = 0$ necessarily. In other words, we have to show that $P(\lambda)$ in (95) and (123) and the λ-polynomials resulting from (93) all vanish identically. We have besides to consider the more general case where $t = sn + 1$, $s > 1$.

22. We shall consider the former question first. It should be observed that $P(\lambda)$ in (95) and (123), and the other λ-polynomials we have presumed to vanish are independent of K_{n+1} or any K of higher order. Hence, if it can be shown that the function

$$y = z + z^{n+1}; \qquad n > 2 \tag{126}$$

cannot be factored into two transformations of period n, then it would follow that $P(\lambda) = 0$ identically, and that the other λ-polynomials vanish correspondingly.

23. We have seen, Theorem V, that if in (36)

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, 3, \cdots,$$

then there is one and only one transformation (36) which will put (34) into the form (35). Suppose now that F in (43) takes the form (126) and we require f and g as defined in (43) to satisfy

$$g[f(z)] = z + z^{n+1}. \tag{127}$$

As we have seen, we must have $\epsilon_1\epsilon_2 = 1$, and $m = n$. Furthermore, from (43) and (127) we observe, on writing out the detailed conditions on λ's and μ's, that each coefficient μ is determined uniquely as a polynomial in λ's of corresponding and lower orders. Hence, *for all orders* $< n + 1$, the coefficients μ have the same relations to the coefficients λ as would be obtained from (43) and the condition

$$g[f(z)] = z. \tag{128}$$

This condition would require $g = f^{-1}$. Furthermore, there exists a unique function

$$h(z) = z + \sum_{s=0}^{\infty} \sum_{r=1}^{n-1} A_{sn+r+1} z^{sn+r+1} \tag{129}$$

such that, symbolically,

$$f = h^{-1}\epsilon_1 h. \tag{130}$$

From (130), $f^{-1} = h^{-1}\epsilon_1^{-1}h$. Hence $g(z)$ as determined by (128) may be written

$$g = h^{-1}\epsilon_1^{-1}h. \tag{131}$$

The first n coefficients of g as determined by (131) are identical with the corresponding coefficients as determined by (127). Furthermore, since f and g in (127) are periodic functions of order n, the coefficients λ_{n+1} and μ_{n+1} are determined necessarily as polynomials of the coefficients of lower order. Furthermore, the conditions are identical with those arising from (130) and (131) for λ_{n+1} and μ_{n+1} respectively. Hence the forms (130) and (131) for f and g hold for coefficients λ and μ of all orders *up to and inclusive of* $n + 1$. Hence the product is valid for powers of the variable including the $(n + 1)$th. But

$$h^{-1}\epsilon_1 h \cdot h^{-1}\epsilon_1^{-1}h = 1.$$

Hence, necessarily,

$$g[f(z)] = z + (\text{powers of } z > n + 1). \tag{132}$$

Obviously (132) is inconsistent with (127), but results necessarily from the assumption that K_1 in (43) $= 1$, that f and g are of period n and that conditions (125) hold.

Hence $z + z^{n+1}$ cannot be factored into transformations of period n. Hence $P(\lambda)$ in (95) and (123) must $= 0$ identically, and all the other conditions for a necessary relation among the K's must be satisfied identically. Hence

If $z + K_{n+1}z^{n+1} + \cdots$ *is to be factored into two transformations of period* n, *we have as a necessary condition*

$$K_{n+1} = 0.$$

24. Suppose now in the discussion preceding Section 18, we choose $t = 2n + 1$. Then t is *odd*. Furthermore, only the coefficients of λ's and μ's of order $n + 1$ in (74), (82) and (76) will be affected. All coefficients λ and μ between the orders $n + 1$ and $2n + 1$ remain independent. The method of Sections 17 and 19 then holds valid so long as

$$k < \frac{t-1}{2}, \quad \text{or} \quad k < n,$$

yielding the necessary conditions

$$K_2 = K_3 = \cdots = K_n = 0.$$

From the previous section this requires in any case

$$K_{n+1} = 0,$$

determining μ_{n+1} in terms of λ's of corresponding and lower orders, hence in terms of λ's of order $< n + 1$. The resulting relation among the K's becomes

$$K_{2n+1} + H_2K_{2n} + \cdots + H_nK_{n+2} = P(\lambda), \tag{133}$$

where $P(\lambda)$ is a polynomial in λ's of order $< n + 1$, replacing λ_{n+1} by its equivalent in terms of λ's of lower order. Hence if

$$K_{n+2} = K_{n+3} = \cdots = K_{2n} = 0,$$

and $P(\lambda)$ vanishes identically, we obtain the necessary condition

$$K_{2n+1} = 0. \tag{134}$$

This presupposes, too, that the λ-polynomials occurring in the process of obtaining (133) all vanish identically. Hence if

$$K_2 = K_3 = \cdots = K_{2n} = 0, \tag{135}$$

then $K_{2n+1} = 0$.

Following identically the same reasoning as above we have for $t = 3n + 1$, if (135) hold true,

$$K_t + H_2K_{t-1} + \cdots + H_nK_{t-n+1} = P(\lambda), \tag{136}$$

where $P(\lambda)$ is some polynomial in λ's of order $< t - n$. Hence again, if $P(\lambda) = 0$ and $K_{t-n+1} = \cdots = K_{t-1} = 0$, we must have $K_t = 0$.

The reasoning is clearly general, and we may put $t = sn + 1$ in (136). Hence if

$$K_2 = K_3 = \cdots = K_{sn} = 0, \tag{137}$$

we must have as a necessary condition

$$K_{sn+1} = 0. \tag{138}$$

Furthermore, we observe that the reasoning of Section 23 is general, and that if (137) hold true, then (138) must be satisfied. This is, furthermore, the only way in which a necessary condition may be required among the coefficients K.

We may now state the complete

THEOREM VI. *If the transformation*

$$F(z) = K_1z + K_2z^2 + \cdots$$

is to be factorable into two periodic transformations, we must have

$$|K_1| = 1,$$

with a commensurable argument; if $K_1 \neq 1$, *the factorization is always possible, the period being so taken that* $K_1 =$ *the product of the leading coeffi-*

cients of the factor transformations; if $K_1 = 1$, *the periods of the factor transformations, if they exist, must be equal; if* $K_1 = 1$, *and* $K_2 \neq 0$, *then* $F(z)$ *is always factorable into two transformations of arbitrary period* > 2; *if* $K_1 = 1$ *and*

$$K_2 = K_3 = \cdots = K_r = 0; \qquad K_{r+1} \neq 0,$$

then $F(z)$ *can not be factored into transformations of order* r *or any factor of* r, *but can always be factored into transformations of any other order* > 2.

25. For example, if

$$F(z) = z + z^{13},$$

$F(z)$ cannot be factored into transformation of period 2,* 3, 4, 6 or 12. It can, however, be factored into transformations of period 5, 7, 8, 9, 10, 11, or any period > 12. On the other hand,

$$F(z) = -z + z^{13}$$

can be factored as the product of transformations of any even period > 2.

It may be observed, also, that in any case where factorization is possible, transformations with equal irreducible periods may be chosen. This is evident from the single restriction (48).

All transformations

$$F(z) = \epsilon z + K_2 z^2 + K_3 z^3 + \cdots; \qquad \epsilon^n = 1$$

are factorable into periodic transformations and constitute a group, the group generated by all periodic transformations. The class defined by $\epsilon = 1$ constitutes a subgroup. The class defined by $\epsilon = 1$, $K_2 = 0$ is a subgroup of the last. The class defined by $\epsilon = 1$, $K_2 = K_3 = 0$ is a subgroup of the previous, and so on. In any of the previous cases, the class defined by $\epsilon = 1$,

$$K_2 = K_3 = \cdots = K_r = 0 \qquad \text{and} \qquad K_{r+1} \neq 0$$

constitutes a subgroup. The latter may be characterized by the index of the highest period of transformations (r) into which $F(z)$ cannot be factored.

It should be observed further that, though

$$F(z) = z + K_{r+1} z^{r+1} + \cdots; \qquad r > 2 \tag{139}$$

cannot be factored into *two* transformations of period r, it can always be factored into *three* transformations of period r. Further, (139) can always be factored into a rotation through a rational angle and two periodic transformations. The latter, too, can be taken of period r, by choosing the corresponding angle.

* See Kasner, loc. cit.

REVERSE CONFORMAL TRANSFORMATIONS (CONFORMAL SYMMETRIES).

26. Kasner also considers another type of conformal transformation. This he calls the *reverse* or *improper* type of transformation, which he defines by

$$y = f(z_0), \tag{140}$$

where f is non-singular at the origin and z_0 is the conjugate of the variable z. He defines reverse transformations of period 2 as *conformal symmetries* since they are conformally reducible to the form $y = z_0$ (conformally equivalent to Schwarzian reflection) and discusses them in parallel with what he terms the *direct conformal transformations.*

We shall now show that no other types of reverse conformal transformations of regular period exist; in other words, every periodic reverse conformal transformation is of irreducible period 2, that is, a conformal symmetry.

27. Denoting the transform of z, $f(z_0)$, by $F(z)$, so that

$$F(z) = f(z_0), \tag{141}$$

we have

$$F_2(z) = f[f(z_0)]_0,$$

where the zero subscript denotes conjugate values.

We now observe that, in general,

$$\begin{gathered}(x+y)_0 = x_0 + y_0, \quad (xy)_0 = x_0 y_0, \quad (z^n)_0 = (z_0)^n, \quad (az^n)_0 = a_0 z_0^n, \\ [f(z)]_0 = f_0(z_0), \quad (z_0)_0 = z, \quad [f(z_0)]_0 = f_0(z),\end{gathered} \tag{142}$$

where the coefficients of f_0 are the conjugates of those of f.

Hence

$$F_2(z) = f[f_0(z)]. \tag{143}$$

Similarly,

$$\begin{aligned}F_3(z) &= f\{f[f_0(z)]\}_0 \\ &= f\{f_0[f_0(z)]_0\} \\ &= f\{f_0[f(z_0)]\}.\end{aligned}$$

Hence we may write

$$F_{2k-1}(z) = (ff_0)^{k-1} f(z_0), \tag{144}$$

$$F_{2k}(z) = (ff_0)^k(z). \tag{145}$$

28. We shall consider the case $F_{2k-1}(z) = z$ first. If f and f_0 be defined by

$$\begin{aligned}f(z) &= a_1 z + a_2 z^2 + \cdots, \\ f_0(z) &= b_1 z + b_2 z^2 + \cdots,\end{aligned} \tag{146}$$

we must have

$$b_r = \frac{|a_r|^2}{a_r}, \qquad |a_1| = 1. \tag{147}$$

Hence $a_1 b_1 = 1$, and

$$f[f_0(z)] = z + \text{(higher powers)}. \tag{148}$$

Hence from (144) we would have

$$a_1 z_0 + \text{(higher powers of } z_0) = z, \tag{149}$$

for all values of z. On comparing coefficients of real and imaginary components of the variable z, the relation (149) is manifestly impossible. Hence, if

$$F_n(z) = z \tag{150}$$

identically, n must be *even*.

29. Suppose now $F_{2k}(z)$ is identically z in (145). Then

$$(ff_0)^k(z) = z. \tag{151}$$

Here ff_0 is a definite transformation. Putting $g(z) = ff_0(z)$, we have then g a *direct* conformal transformation of period k, and we may write

$$g(z) = ff_0(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots; \qquad \epsilon^k = 1. \tag{152}$$

But by (148) the leading coefficient of $ff_0(z)$ is 1. Hence in (152) we must have $\epsilon = 1$. Hence, by Theorem I,

$$\lambda_2 = \lambda_3 = \cdots = 0,$$

and we have

$$F_2(z) = f[f_0(z)] = z \tag{153}$$

necessarily. Hence in any case F is of period 2. Hence we may state

THEOREM VII. *Every periodic reverse conformal transformation is of period 2; in other words, the only kind of periodic reverse conformal transformations are conformal symmetries.*

30. In his discussion of conformal symmetries, Kasner obtains the result that the transformation

$$K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \qquad |K_1| = 1, \tag{154}$$

is always factorable into two symmetries if the angle of K_1 is *irrational*. From the discussion in this paper, (154) is factorable into two direct periodic conformal transformations if the angle of K_1 is *rational*. Hence we have

THEOREM VIII. *The transformation*

$$K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \qquad |K_1| = 1,$$

is always factorable either into two conformal symmetries or into two direct periodic conformal transformations.

It would seem also that, in general, a direct periodic conformal transformation is not factorable into two symmetries.

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